0.1

Fix integers $k < n$ and an $(n-k) \times k$ integer matrix C. Let a torus T^k act on $V \cong \mathbb{C}^n$ with weights

$$t_1, \dots, t_k, \prod t_i^{C_{1i}}, \dots, \prod t_i^{C_{n-k,i}}$$

where $(t_1, \dots, t_k) \in T^k \cong (\mathbb{C}^\times)^k$. Let

$$X = T^*V /\!\!/\!\!/\!\!/ \, T^k$$

be the algebraic symplectic reduction of T^*V by T^k with respect to some fixed stability parameter $\theta \in \mathbb{R}^k$.

On X, there is a natural action of

$$T^{n-k} \times \mathbb{C}^\times_\hbar \cong (\mathbb{C}^\times)^{n-k} \times \mathbb{C}^\times \ni (a_1, \dots, a_{n-k}, \hbar)$$

where a_i scales the last $(n-k)$ coordinates of $\mathbb{C}^n$ while $\hbar$ scales the cotangent directions. The point

$$x = (1^k, 0^{n-k}) \in V \subset T^*V$$

projects onto a torus-fixed point of X which we will denote by the same symbol. We begin by writing the 1-leg vertex corresponding to the point $x \in X$ as a multiple hypergeometric series.

0.2

Consider the space of quasi-maps to X from $B = \mathbb{P}^1$. By definition [], quasi-map from B to a quotient by a group G, here $G = T^k$, is a principal G-bundle P over B together with a section f of the induced bundle

$$\begin{array}{ccc} T^*V & \hookrightarrow & P \times_G T^*V \\ & & \downarrow \uparrow f \\ & & B \end{array} \quad . \tag{1}$$

In our case, because we work with a symplectic reduction, we will need to restrict to the zero locus of the moment map.

Let $d_1, \dots, d_k$ be the degrees of P. Then

$$P \times_G T^*V = \bigoplus_i \mathscr{O}(d_i) \oplus \bigoplus_i \mathscr{O}(-d_i) \otimes \hbar \oplus \\ \bigoplus_j \mathscr{O}\left(\sum C_{ji} d_i\right) \otimes a_i \oplus \bigoplus_j \mathscr{O}\left(-\sum C_{ji} d_i\right) \otimes \hbar / a_i \tag{2}$$

as an equivariant bundle. An additional torus $\mathbb{C}^\times_q$ acts on quasimaps scaling the source B with weight $q \in \mathbb{C}^\times_q$ at the fixed point $0 \in \mathbb{P}^1$.

A torus fixed quasimap f that evaluates to x at infinity of B is necessarily given by nonzero sections of $\bigoplus_i \mathscr{O}(d_i)$ with a d_i-fold zero at $0 \in B$. Its existence requires $d_i \geq 0$. We denote such unique torus-fixed quasimap of given degrees d_i by f_d.

0.3

The deformations of the map f_d are the following. First, each of the sections of $\mathscr{O}(d_i)$ may be deformed modulo an overall scaling (automorphism of the source principal bundle), which gives the characters

$$-1 + \operatorname{char} H^\bullet(\mathscr{O}(d_i)) = q + q^2 + \cdots + q^{d_i}, \quad i = 1, \ldots, k\,.$$

The dual obstruction weights come from

$$-\hbar + \operatorname{char} H^\bullet(\mathscr{O}(-d_i) \otimes \hbar) = -\hbar(1 + q^{-1} + \cdots + q^{1-d_i})\,,$$

where the $-\hbar$ term comes from the moment map equation. Analogously, subtracting the tangent weights at $x \in X$, we get

$$-a_i + \operatorname{char} H^\bullet(\mathscr{O}(\textstyle\sum C_{ji} d_i) \otimes a_i) = a_i \dfrac{q^{\sum C_{ji} d_i} - 1}{1 - q^{-1}}$$

and similarly for the dual term. Note that the character of the virtual tangent space at f_d is self-dual

$$\operatorname{char} \operatorname{Obs}(f_d) = \kappa \otimes \overline{\operatorname{char} \operatorname{Def}(f_d)}\,, \quad \kappa = \hbar q\,.$$

Since the product of 5 equivariant parameters in 5-dimensions equals 1, the duality works as follows

$$(\hbar, \kappa) \xrightarrow{\text{duality}} (1/\kappa, 1/\hbar)\,, \tag{3}$$

which fixes the ratio $q = \kappa/\hbar$.

0.4

Together with the square roots of the virtual canonical bundle, the weight of the deformation theory of f_d equals

$$\prod_w \frac{(\kappa/w)^{1/2} - (w/\kappa)^{1/2}}{w^{1/2} - w^{-1/2}} = \prod_w (-\kappa^{1/2}) \frac{1 - w/\kappa}{1 - w}\,,$$

where the product is over all weights in the deformation space $\operatorname{Def}(f_d)$ of f_d.

We shift the Kähler parameters so that the contribution of degree d is weighted by $\prod_i ((-\kappa^{-1/2})^{1+\sum C_{ji}} Q_i)^{d_i}$, this is a generalization of the shift by $i\pi c_1(\mathscr{K}_V)$ in $H^2(X, \mathbb{C})$. Then we get the following formula for the vertex

$$\mathsf{V} = \sum_{d_1, \ldots, d_k \geq 0} Q^d \prod_{i=1}^{k} \frac{(1/\hbar)_{d_i}}{(q)_{d_i}} \prod_{j=1}^{n-k} \frac{(a_j/\kappa)_{(Cd)_j}}{(a_j)_{(Cd)_j}}\,,$$

where

$$(z)_k = \frac{(z)_\infty}{(q^k z)_\infty}\,, \quad (z)_\infty = \phi(z) = \prod_{n \geq 0} (1 - q^n z)$$

and where $(Cd)_j = \sum C_{ji} d_i$ is the jth entry of the matrix product Cd.

0.5

It is better to replace the vertex V by the following series

$$\widetilde{\mathsf{V}} = \prod_{j=1}^{n-k} \frac{(a_j/\kappa)_\infty}{(a_j)_\infty} \mathsf{V} \tag{4}$$

$$= \sum_{d_1,\dots,d_k \geq 0} Q^d \prod_{i=1}^{k} \frac{(1/\hbar)_{d_i}}{(q)_{d_i}} \prod_{j=1}^{n-k} \frac{(a_j q^{(Cd)_j})_\infty}{(a_j q^{(Cd)_j}/\kappa)_\infty} \,. \tag{5}$$

The prefactor in (4) should have a natural interpretation in terms of replacing $B = \mathbb{P}^1$ by $B = \mathbb{C}$ and regularizing the corresponding infinite product.

0.6

Now recall the q-binomial theorem that says

$$\frac{(bz)_\infty}{(z)_\infty} = \sum_m \frac{(b)_m}{(q)_m} z^m \,.$$

It implies

$$\frac{(a_j q^{(Cd)_j})_\infty}{(a_j q^{(Cd)_j}/\kappa)_\infty} = \sum_m \frac{(\kappa)_m}{(q)_m} (a_j/\kappa)^m q^{m(Cd)_j}$$

and therefore

$$\widetilde{\mathsf{V}} = \sum_{d,m} Q^d (a/\kappa)^m q^{m^T Cd} \prod_{i=1}^{k} \frac{(1/\hbar)_{d_i}}{(q)_{d_i}} \prod_{j=1}^{n-k} \frac{(\kappa)_{m_j}}{(q)_{m_j}} \,.$$

This is manifestly symmetric under (3) and moreover, this is a result of applying an operator which looks like an abelian R-matrix to a fully factored expression

$$\widetilde{\mathsf{V}} = q^\Delta \prod_{i=1}^{k} \frac{(Q_i \hbar)_\infty}{(Q_i)_\infty} \prod_{j=1}^{n-k} \frac{(a_j)_\infty}{(a_j/\kappa)_\infty}$$

where

$$\Delta = \sum_{ij} C_{ji}\, a_j \frac{\partial}{\partial a_j} \otimes Q_i \frac{\partial}{\partial Q_i}$$

where the $\otimes$ sign is put in purely for emphasis.