Q: When do we have $[M(\lambda):L(\mu)] \neq 0$?

Last time (Verma): It suffices to put $\mu = s_\alpha \bullet \lambda$ s.t. $\lambda \geq \mu$ because then $M(\mu) \hookrightarrow M(\lambda)$.

Cor: if we have $\mu = s_{\alpha_1} \cdots s_{\alpha_r} \bullet \lambda \leq s_{\alpha_2} \cdots s_{\alpha_r} \bullet \lambda \leq \ldots \leq s_{\alpha_r} \bullet \lambda \leq \lambda$ $(*)$
we get $M(\mu) \hookrightarrow M(\lambda)$ also.
When $(*)$ is satisfied we say μ is strongly linked to λ and we write $\mu \uparrow \lambda$

BGG Theorem: If $[M(\lambda):L(\mu)] \neq 0$, then $\mu \uparrow \lambda$.
We will prove this shortly.

Recall that $\mathcal{O}$ is split into categories $\mathcal{O}_\lambda$, indexed by ρ-antidominant weights λ. Assume λ is integral and dot-regular, so in particular λ is the unique ρ-antidominant weight in $W \bullet \lambda$. Then we can rephrase the
$(|W \bullet \lambda| = |W|)$
condition $w' \bullet \lambda \uparrow w \bullet \lambda$ exclusively in terms of the Weyl group.

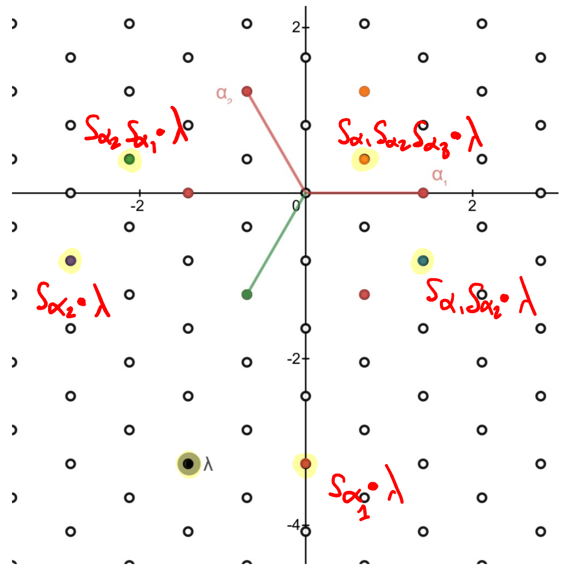

Fig 1: The highlighted dots are the elements of $W \bullet \lambda$
Here λ is the lowest weight in $W \bullet \lambda$ and is therefore ρ-antidominant.

Skip?
Reminder on the Bruhat order: $w' \xrightarrow{\alpha} w$ means $w' = s_\alpha w$ and $\ell(w') = \ell(w) - 1$
$w' \leq w$ means $w' \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} w$

Also $\bullet\ \ell(w) = \min\{k \mid s_{\alpha_1} \cdots s_{\alpha_k} = w\} = \#\{\alpha > 0 : w\alpha < 0\}$ (simple)
$\bullet\ \ell(w^{-1}) = \ell(w)$

Prop: If λ is integral, dot-regular and ρ-antidominant, then
$w' \bullet \lambda \uparrow w \bullet \lambda \Leftrightarrow w' \leq w$

Proof: It suffices to prove $s_\alpha w \uparrow w \bullet \lambda \Leftrightarrow s_\alpha w \xrightarrow{\alpha} w$
These are equivalent since

$$
\begin{aligned}
& s_\alpha (w \bullet \lambda + \rho) < w \bullet \lambda + \rho \\
\Leftrightarrow\ & s_\alpha w (\lambda + \rho) < w(\lambda + \rho) \\
\Leftrightarrow\ & \langle w(\lambda+\rho), \alpha^\vee \rangle > 0 \\
\Leftrightarrow\ & \langle \lambda + \rho, w^{-1}\alpha^\vee \rangle > 0 \\
\overset{\lambda \text{ antidom}}{\Leftrightarrow}\ & w^{-1}\alpha < 0 \\
\Leftrightarrow\ & w^{-1} s_\alpha \alpha > 0 \\
\Leftrightarrow\ & \ell(w^{-1} s_\alpha) = \ell(w^{-1}) - 1 \\
\Leftrightarrow\ & \ell(s_\alpha w) = \ell(w) - 1 \\
\Leftrightarrow\ & s_\alpha w \xrightarrow{\alpha} w \qquad \square
\end{aligned}
$$

Rmk: The condition $w' \leq w$ is independent of λ! (so we may only look at $\mathcal{O}_0$)

Summary: $[M(w \bullet \lambda) : L(w' \bullet \lambda)] \neq 0 \Leftrightarrow w' \bullet \lambda \uparrow w \bullet \lambda \Leftrightarrow w' \leq w$.
(if λ dot-reg, ρ-antidom, integral)

Rmk: it is possible to have $w \cdot \lambda < \lambda$ yet $w \cdot \lambda \not\uparrow \lambda$ (only in rank ≥ 3)
Example: $\mathfrak{g} = \mathfrak{sl}_4\mathbb{C}$, $\lambda = s_{\alpha_2} s_{\alpha_3} \cdot (-\lambda_2)$. Computation shows $\lambda - \mu = \alpha_1 + \alpha_3$. However,
$\mu = s_{\alpha_3} s_{\alpha_2} s_{\alpha_1} \cdot (-\lambda_2)$

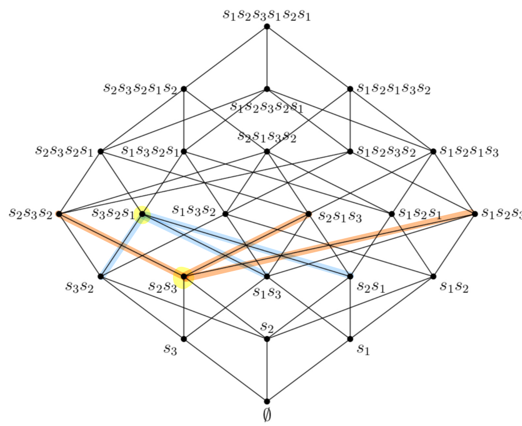

Fig 2: $s_{\alpha_2}s_{\alpha_3}$ and $s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}$ are unrelated (Bruhat order)

The BGG Theorem is a corollary of:

Theorem (Jantzen Filtration): let $\lambda \in \mathfrak{h}^*$ arbitrary. Then $M(\lambda)$ has a filtration
$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset \dots$ $\quad (M(\lambda)^{\gg 0} = 0)$
such that
(a) Each nonzero quotient $M(\lambda)^i / M(\lambda)^{i+1}$ has a nondegenerate contravariant form

(b) $M(\lambda) / M(\lambda)^1 = L(\lambda)$ $\quad$ (i.e. $M(\lambda)^1 = N(\lambda)$)

(c) $\displaystyle\sum_{i>0} \operatorname{ch} M(\lambda)^i = \sum_{\alpha>0,\ s_\alpha \cdot \lambda < \lambda} \operatorname{ch} M(s_\alpha \cdot \lambda)$ $\quad$ ("Jantzen sum formula")

Cor: (BGG Theorem): By induction on $k = \#\{\mu \in W \cdot \lambda \mid \mu \leq \lambda\}$. If $k = 1$ then λ is minimal so $M(\lambda)$ is simple ✓
Otherwise assume $[M(\lambda) : L(\mu)] > 0$, $\mu < \lambda$. Then $[M(\lambda)^1 : L(\mu)] > 0$ so $[M(s_\alpha \cdot \lambda) : L(\mu)] > 0$
for some $\alpha > 0$ s.t. $s_\alpha \cdot \lambda < \lambda$, so $s_\alpha \cdot \lambda \uparrow \lambda$. But by induction hypothesis $\mu \uparrow s_\alpha \cdot \lambda$, so $\mu \uparrow \lambda$. ∎

Application of Jantzen's Filtration

For the next application of the theorem, we need a result that was mentioned last time:
(4.10)
Prop: let λ be integral, dot-regular, ρ-antidominant. Then for any $w \cdot \lambda \in W \cdot \lambda$, $M(\lambda) = L(\lambda)$ is the unique simple submodule of $M(w \cdot \lambda)$. Furthermore, $[M(w \cdot \lambda) : L(\lambda)] = 1$.
Proof: We saw the first part last time.
For the second part, it suffices to prove, by BGG Reciprocity, that $(P(\lambda) : M(w \cdot \lambda)) \leq 1$
Now, recall that $-\rho$ is ρ-dominant $\Rightarrow$ $M(-\rho)$ is projective. Also $w_0 \cdot \lambda$ is ρ-dominant so $w_0 \cdot \lambda + \rho \in \Lambda^+$.
Thus $L(w_0 \cdot \lambda + \rho)$ is f.dim'l hence $M = M(-\rho) \otimes L(w_0 \cdot \lambda + \rho)$ is projective.

Claim: M has a standard filtration with $M(w\cdot\lambda)$ appearing at most once and $M(\lambda)$ appearing only at the top

Using the claim: since $M \twoheadrightarrow M(\lambda)$ we have $P(\lambda) \underset{\oplus}{\subseteq} T$ (since T is projective) and so $P(\lambda)$ has a std filtration with $M(w\cdot\lambda)$ appearing at most once, as desired.

Proof of the claim: let $v_1, \dots, v_n$ be a basis of weight vectors for $L(w_0\cdot\lambda+\rho)$ with weights $\mu_1, \dots, \mu_n$. Reorder the basis so that $\mu_i \leq \mu_j \Rightarrow i \leq j$ and so that v_1 has weight λ (which is minimal since $w_0(w_0\cdot\lambda+\rho) = \lambda+\rho$)

Now tensor the filtration of $U(\mathfrak{b})$-modules

$$\mathbb{C}_{-\rho} \otimes \mathrm{Span}_{U(\mathfrak{b})}(v_n) \subsetneq \mathbb{C}_{-\rho} \otimes \mathrm{Span}_{U(\mathfrak{b})}(v_n, v_{n-1}) \subsetneq \dots \subsetneq \mathbb{C}_{-\rho} \otimes L(w_0\cdot\lambda+\rho)$$

(with quotients $\mathbb{C}_{-\rho} \otimes \mathbb{C}_{\mu_i}$
Note: $w(\lambda+\rho)$ appears only once)

with $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} -$ (which is exact of f.diml $U(\mathfrak{b})$-modules). We get

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_{-\rho} \otimes \mathrm{Span}_{U(\mathfrak{b})}(v_n)) \subset \dots \subset U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_{-\rho} \otimes L(w_0\cdot\lambda+\rho))$$

$$= (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-\rho}) \otimes L(w_0\cdot\lambda+\rho)$$

$$= M(-\rho) \otimes L(w_0\cdot\lambda+\rho)$$

and each quotient is isomorphic to $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_{-\rho} \otimes \mathbb{C}_{\mu_i}) = M(\mu_i - \rho)$, in particular $M(\mu_1 - \rho) = M(\lambda)$.

Break?

Finding the composition factors of $M(\lambda)$ (λ integral, dot-reg) for $\mathfrak{sl}_3\mathbb{C}$

Recall that Knowing $\{\mathrm{ch}\, M(w\cdot\lambda)\}_{w\in W}$ in terms of $\{\mathrm{ch}\, L(w\cdot\lambda)\}_{w\in W}$ (or vice-versa)
$\Leftrightarrow$ Knowing $[M(w\cdot\lambda) : L(w'\cdot\lambda)]$ $\forall w,w'\in W$.

$\mathfrak{g}=\mathfrak{sl}_3\mathbb{C}$, λ integral, dot-regular, ρ-antidominant (e.g. $\lambda=-2\rho$)

So far we have $\mathrm{ch}\, M(\lambda) = \mathrm{ch}\, L(\lambda)$ (since λ ρ-antidom, $M(\lambda)$ simple)
Next, $[M(s_{\alpha_1}\cdot\lambda) : L(s_{\alpha_1}\cdot\lambda)] = 1$
$[M(s_{\alpha_1}\cdot\lambda) : L(\lambda)] = 1$ (by the Proposition)
and these are all the candidates, so

$$\mathrm{ch}\, M(s_{\alpha_1}\cdot\lambda) = \mathrm{ch}\, L(\lambda) + \mathrm{ch}\, L(s_{\alpha_1}\cdot\lambda)$$
sim: $$\mathrm{ch}\, M(s_{\alpha_2}\cdot\lambda) = \mathrm{ch}\, L(\lambda) + \mathrm{ch}\, L(s_{\alpha_2}\cdot\lambda)$$

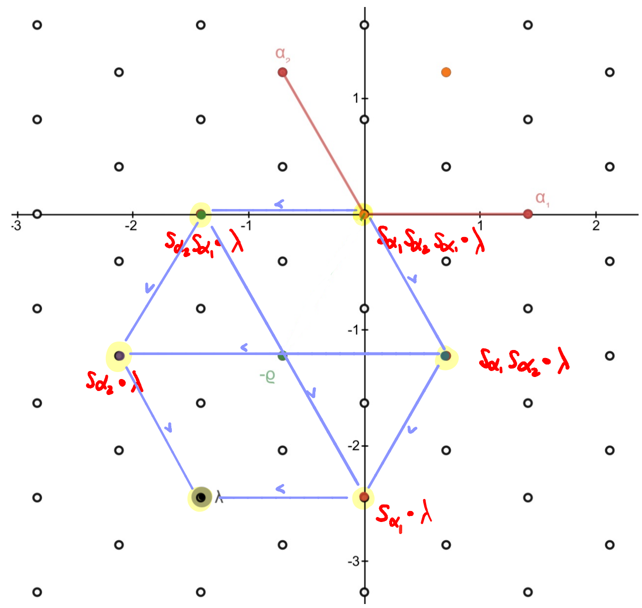

Fig 2: The orbit $W\cdot\lambda$. Here $\mu_1 \xleftarrow{<} \mu_2$ means $\mu_1 \uparrow \mu_2$

Next, for $s_{\alpha_1}s_{\alpha_2}\cdot\lambda$ we know
$[M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) : L(s_{\alpha_1}s_{\alpha_2}\cdot\lambda)] = 1 = [M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) : L(\lambda)]$
but we need
$[M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) : L(s_{\alpha_1}\cdot\lambda)]$, $[M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) : L(s_{\alpha_2}\cdot\lambda)]$

Jantzen's sum formula
$$\Rightarrow \sum_{i>0} \mathrm{ch}\, M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda)^i = \mathrm{ch}\, M(s_{\alpha_1}\cdot\lambda) + \mathrm{ch}\, M(s_{\alpha_2}\cdot\lambda)$$
$$= \mathrm{ch}\, L(s_{\alpha_1}\cdot\lambda) + \mathrm{ch}\, L(s_{\alpha_2}\cdot\lambda) + 2\, L(\lambda)$$
Since $[M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) : L(\lambda)] = 1$, we deduce $M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda)^2 = L(\lambda)$ and
$$\mathrm{ch}\, M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda)^1 = \mathrm{ch}\, L(s_{\alpha_1}\cdot\lambda) + \mathrm{ch}\, L(s_{\alpha_2}\cdot\lambda) + \mathrm{ch}\, L(\lambda)$$

Therefore, $[M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) : L(s_{\alpha_1}\cdot\lambda)] = 1 = [M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) : L(s_{\alpha_2}\cdot\lambda)]$
sim: $[M(s_{\alpha_2}s_{\alpha_1}\cdot\lambda) : L(s_{\alpha_1}\cdot\lambda)] = 1 = [M(s_{\alpha_2}s_{\alpha_1}\cdot\lambda) : L(s_{\alpha_1}\cdot\lambda)]$

Finally, for $s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}\cdot\lambda = w_0\cdot\lambda$, we have

$$\sum_{i>0} \mathrm{ch}\, M(w_0\cdot\lambda)^i = \mathrm{ch}\, M(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) + \mathrm{ch}\, M(s_{\alpha_2}s_{\alpha_1}\cdot\lambda) + \mathrm{ch}\, L(\lambda)$$
$$= \mathrm{ch}\, L(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) + \mathrm{ch}\, L(s_{\alpha_2}s_{\alpha_1}\cdot\lambda) + 2(\mathrm{ch}\, L(s_{\alpha_1}\cdot\lambda) + \mathrm{ch}\, L(s_{\alpha_2}\cdot\lambda)) + 3\,\mathrm{ch}\, L(\lambda)$$

Again, since $[M(w_0\cdot\lambda) : L(\lambda)] = 1$, we deduce $M(w_0\cdot\lambda)^3 = L(\lambda)$
Two possibilities remain:

$$\begin{cases} \mathrm{ch}\, M(w_0\cdot\lambda)^2 = \mathrm{ch}\, L(s_{\alpha_1}\cdot\lambda) + \mathrm{ch}\, L(s_{\alpha_2}\cdot\lambda) + \mathrm{ch}\, L(\lambda) \\ \mathrm{ch}\, M(w_0\cdot\lambda)^1 = \mathrm{ch}\, L(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) + \mathrm{ch}\, L(s_{\alpha_2}s_{\alpha_1}\cdot\lambda) + 1\,\mathrm{ch}\, L(s_{\alpha_1}\cdot\lambda) + 1\,\mathrm{ch}\, L(s_{\alpha_2}\cdot\lambda) + \mathrm{ch}\, L(\lambda) \end{cases}$$

2 0 1 0 0

or

$$\begin{cases} \operatorname{ch} M(w_0\cdot\lambda)^2 = \operatorname{ch} L(\lambda) \\ \operatorname{ch} M(w_0\cdot\lambda)^1 = \operatorname{ch} L(s_{\alpha_1}s_{\alpha_2}\cdot\lambda) + \operatorname{ch} L(s_{\alpha_2}s_{\alpha_1}\cdot\lambda) + 2\operatorname{ch} L(s_{\alpha_1}\cdot\lambda) + 2\operatorname{ch} L(s_{\alpha_2}\cdot\lambda) + \operatorname{ch} L(\lambda) \end{cases} \qquad (*)$$

Humphreys claims this case can be dealt with using the same tools as in the previous cases, but I don't see how. Instead, here is an ad hoc argument: the weight space $M(w_0\cdot\lambda)_{s_{\alpha_1}\cdot\lambda}$ has dimension 2 and

$L(s_{\alpha_1}\cdot\lambda)_{s_{\alpha_1}\cdot\lambda} = 1, \quad L(s_{\alpha_2}s_{\alpha_1}\cdot\lambda)_{s_{\alpha_1}\cdot\lambda} = 1$ (since $M(s_{\alpha_2}s_{\alpha_1}\cdot\lambda)_{s_{\alpha_1}\cdot\lambda} = 2$ and we know $\operatorname{ch} M(s_{\alpha_2}s_{\alpha_1}\cdot\lambda)$)

so $[M(w_0\cdot\lambda) : L(s_{\alpha_1}\cdot\lambda)] \leq 1$

We thus discard (*) and conclude $[M(w_0\cdot\lambda) : L(w\cdot\lambda)] = 1 \quad \forall w \in W$

Incidentally, this provides an example of the question we discussed last week:

$$\operatorname{soc}\left(M(0)/L(-2\rho)\right) \supset M(0)^2/L(-2\rho) = L(s_{\alpha_1}\cdot(-2\rho)) \oplus L(s_{\alpha_2}\cdot(-2\rho)) = L(-\rho-\alpha_1) \oplus L(-\rho-\alpha_2)$$

Skip: (Proof of $\oplus$: let $\varphi \in \operatorname{Out}(\mathfrak{sl}_3\mathbb{C})$ corresponding to $\underset{\alpha_1}{\bullet}\!-\!\underset{\alpha_2}{\bullet}$. Then if $M = M(0)/L(-2\rho)$, set $M^\varphi = M$, action twisted by φ. This is isomorphic to M since $\varphi(0) = 0$,

However $L(-\rho-\alpha_1)^\varphi = L(-\rho-\alpha_2)$ so one can't be on top of the other)

Proof of Jantzen's Filtration Theorem

"Key Lemma"

Let A be a PID and $M = A^r$ with a nondegenerate bilinear form $(\,,\,)$ with determinant $D \neq 0$.
Let $p \in A$ be prime. Define a filtration $M = M(0) \supset M(1) \supset \dots$ by $M(n) = \{m \in M : (m, M) \subset p^n A\}$
Write also $\bar{A} = A/_{pA}$, $\bar{M} = M/_{pM}$ etc.
Then

(a) $v_p(D) = \sum_{n>0} \dim_{\bar{A}} \overline{M(n)}$ $\quad$ ($\overline{M(n)} = 0$ for n large)

(b) $(\,,\,)_n = p^{-n}(\,,\,)$ induces a nondeg form on $\overline{M(n)} / \overline{M(n+1)}$

Skip?: Proof omitted. Example instead: $A = \mathbb{C}[T]$, $M = AX \oplus AY$, bilinear form $\begin{pmatrix} 2T & 0 \\ 0 & T^3 \end{pmatrix}$
Then

$M(0) = AX \oplus AY$	$\overline{M(0)} = \mathbb{C}X \oplus \mathbb{C}Y$
$M(1) = AX \oplus AY$	$\overline{M(1)} = \mathbb{C}X \oplus \mathbb{C}Y$
$M(2) = (T)X \oplus AY$	$\overline{M(2)} = \mathbb{C}Y$
$M(3) = (T^2)X \oplus AY$	$\overline{M(3)} = \mathbb{C}Y$
$M(4) = (T^3)X \oplus (T)Y$	$\overline{M(4)} = 0$

$\Big\}\ 4 = v_T(T^4)$ and eg $\overline{M(1)} / \overline{M(2)}$ has form (2)

Theorem (Jantzen Filtration): Let $\lambda \in \mathfrak{h}^*$ arbitrary. Then $M(\lambda)$ has a filtration
$$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset \dots \supset M(\lambda)^N = 0$$
such that

(a) Each nonzero quotient $M(\lambda)^i / M(\lambda)^{i+1}$ has a nondegenerate contravariant form

(b) $M(\lambda) / M(\lambda)^1 = L(\lambda)$ $\quad$ (i.e. $M(\lambda)^1 = N(\lambda)$)

(c) $\sum_{i>0} \mathrm{ch}\, M(\lambda)^i = \sum_{\alpha>0,\, s_\alpha \cdot \lambda < \lambda} \mathrm{ch}\, M(s_\alpha \cdot \lambda)$ $\quad$ ("Jantzen sum formula")

We need to define $M(\lambda)^i$. The idea is to introduce a free variable T by setting $A = \mathbb{C}[T]$, $K = \mathbb{C}(T)$ and $\mathfrak{g}_K := K \otimes_{\mathbb{C}} \mathfrak{g}$, $\mathfrak{g}_A = A \otimes_{\mathbb{C}} \mathfrak{g}$. Let $\lambda_T := \lambda + T\rho \in \mathfrak{h}^*_K$.

The theory over K is the same as over $\mathbb{C}$, and since $\langle \lambda_T + \rho, \alpha^\vee \rangle \notin \mathbb{Z}\ \forall \alpha \in \Phi^+$, $\lambda_T + \rho$ is antidominant and therefore $M(\lambda_T)$ is simple, so the contravariant form on it is nondegenerate.
Also, the weight spaces are $M(\lambda_T)_{\lambda_T - \nu}$ for $\nu \in R^+$

We also have an A-form $M(\lambda_T)_A \subset M(\lambda_T)$. Write $M_{\lambda_T - \nu} = M(\lambda_T)_{\lambda_T - \nu} \cap M(\lambda_T)_A$

Finally, set $M(\lambda_T)^i_A := \sum_{\nu \in R^+} M_{\lambda_T - \nu}(i)$

To get the filtration for $M(\lambda)$, set $T = 0$ i.e. $M(\lambda)^i = M(\lambda_T)^i_A / T M(\lambda_T)^i_A$

The Key lemma tells us $T^{-i}(\,,\,)$ induces a nondeg contravariant form on $M(\lambda)^i / M(\lambda)^{i+1}$.
Since $M(\lambda)/M(\lambda)^1$ is also a h.w. module, it must be simple. This proves (a) and (b).

Example of this construction: $\mathfrak{sl}_3\mathbb{C}$, $\lambda = -\lambda_1$. Then $\lambda_T = -\lambda_1 + T\rho$.

We look at a single weight space $M_{\lambda_T-\alpha_1-\alpha_2}$. Now $M_{\lambda_T-\alpha_1-\alpha_2}$ has two basis vectors: $a = f_{\alpha_1}f_{\alpha_2}v^+$ and $b = f_{\alpha_1+\alpha_2}v^+$

The contravariant form wrt this basis is $\begin{pmatrix} T^2 & -T \\ -T & 2T-1 \end{pmatrix}$

For instance, $(f_{\alpha_1}f_{\alpha_2}v^+, f_{\alpha_1}f_{\alpha_2}v^+) = (v^+, e_{\alpha_2}e_{\alpha_1}f_{\alpha_1}f_{\alpha_2}v^+)$.

$$e_{\alpha_2}e_{\alpha_1}f_{\alpha_1}f_{\alpha_2}v^+ = e_{\alpha_2}f_{\alpha_1}\underbrace{e_{\alpha_1}f_{\alpha_2}v^+}_{0} + e_{\alpha_2}h_{\alpha_1}f_{\alpha_2}v^+ = e_{\alpha_2}f_{\alpha_2}h_{\alpha_1}v^+ + e_{\alpha_2}f_{\alpha_2}v^+ = (1+h_{\alpha_1})h_{\alpha_2}v^+$$

$$= (1+h_{\alpha_1})\underbrace{(-\lambda_1+T\rho)(\alpha_2)}_{T}v^+ = T\underbrace{(1+(-\lambda_1+T\rho)(\alpha_1))}_{1-1+T}v^+ = T^2v^+.$$

Therefore $M_{\lambda_T-\alpha_1-\alpha_2}(i) = \begin{cases} Aa \oplus Ab & i=0 \\ Aa \oplus (T)b & i=1 \\ Aa \oplus (T^2)b & i=2 \\ (T)a \oplus (T^3)b & i=3 \end{cases}$, $M(\lambda)^i_{\lambda-\alpha_1-\alpha_2} = \begin{cases} \mathbb{C}a \oplus \mathbb{C}b & i=0 \\ \mathbb{C}a & i=1 \\ \mathbb{C}a & i=2 \\ 0 & i\geq 3 \end{cases}$

-

In order to verify (c) $\sum_{i>0} \operatorname{ch} M(\lambda)^i = \sum_{\alpha>0,\, s_\alpha\cdot\lambda<\lambda} \operatorname{ch} M(s_\alpha\cdot\lambda)$, we need to compute $\sum_{i>0} \dim M(\lambda)^i_{\lambda-\nu}$

The Key Lemma tells us that this is $\nu_T(D_\nu(\lambda_T))$ (Here $D_\nu(\lambda_T)$ is the determinant of the form on $M_{\lambda_T-\nu} = M(\lambda_T)_A \cap M(\lambda_T)_{\lambda_T-\nu}$)

Claim: (Shapovalov) $D_\nu(\lambda_T) = \prod_{\alpha>0}\prod_{r>0}(\langle\lambda_T+\rho,\alpha^\vee\rangle - r)^{P(\nu-r\alpha)}$ (Here $P(\gamma) = \#$ ways to write γ as a sum of positive roots)

Using the claim, note $\langle\lambda_T+\rho,\alpha^\vee\rangle - r = \langle\lambda+\rho,\alpha^\vee\rangle - r + T$ is a multiple of T iff $\langle\lambda+\rho,\alpha^\vee\rangle = r$ iff $s_\alpha\cdot\lambda<\lambda$. So $\nu_T(D_\nu(\lambda_T)) = P(\nu - \langle\lambda+\rho,\alpha^\vee\rangle\alpha)$ and therefore

$$\sum_{i>0}\operatorname{ch} M(\lambda)^i = \sum_{\nu\in R^+}\sum_{\substack{\alpha>0,\\ s_\alpha\cdot\lambda<\lambda}} P(\nu - \langle\lambda+\rho,\alpha^\vee\rangle\alpha)e^{\lambda-\nu}$$

$$= \sum_{\substack{\alpha>0\\ s_\alpha\cdot\lambda<\lambda}}\sum_{\nu'\in R^+} P(\nu')e^{\lambda-\langle\lambda+\rho,\alpha^\vee\rangle\alpha-\nu'}$$

$P(\nu-\langle\lambda+\rho,\alpha^\vee\rangle\alpha)\neq 0$ so $\nu' = \nu - \langle\lambda+\rho,\alpha^\vee\rangle\alpha \in R^+$

On the other hand, $\operatorname{ch} M(s_\alpha\cdot\lambda) = \sum_{\nu'\in R^+} P(\nu')e^{s_\alpha\cdot\lambda-\nu'}$, and (c) follows.

Skip? Shapovalov's determinantal formula

In order to get the bilinear form on $M_{\lambda-\nu}$ above, we computed the contravariant form by simplifying the PBW monomials as follows

$(f_i v^+, f_j v^+) = (v^+, \tau(f_i) f_j v^+) = (v^+, t_{ij} v^+)$, where $t_{ij} \in U(\mathfrak{h})$. Then $t_{ij} v^+ = \lambda_T(t_{ij}) v^+$

The determinant of the form is then $\det(\lambda_T(t_{ij}))$, but we can also get this by

$$\det(t_{ij}) v^+ = \det(\lambda_T(t_{ij})) v^+$$

Ultimately, knowing $\det(t_{ij})$ will tell us $D_\nu(\lambda_T)$ for all λ. This is Shapovalov's formula:

$$D_\nu = \det(t_{ij}) = \prod_{\alpha>0} \prod_{r>0} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{P(\nu - r\alpha)}$$

Continuing the previous example, we would have $(t_{ij}) = \begin{pmatrix} \overbrace{h_\beta(h_\alpha+1)}^{\text{already computed, others are easy}} & -h_\beta \\ -h_\beta & h_\alpha + h_\beta \end{pmatrix}$,

whose determinant is $h_\beta h_\alpha (h_\alpha + h_\beta + 1)$

Comment on the proof strategy?

Comment on what happens if we replace ρ by something else?

-