Introduction to Perverse Sheaves

Cailan Li

March 5th, 2020

1 Intersection Homology

The motivation for Intersection Homology is that Poincare duality (and the rest of the Kahler package) fails for singular spaces. For example consider [Draw $X = S^2 \vee S^2$].

By Mayer-Vietoris we have

$$H_2(pt) \to H_2(S^2) \oplus H_2(S^2) \to H_2(X) \to H_1(pt)$$

and so we have that $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ but $H_0(X) = \mathbb{Z}$. However if we resolve the singuarlity we end up with $S^2 \sqcup S^2$ and $H_0(S^2 \sqcup S^2) = \mathbb{Z} \oplus \mathbb{Z}$. In general if $\widetilde{X}$ is the normalization of X (middle-perversity) intersection homology will have the property that

$$IH_*(X, \mathbb{Z}) \cong H_*(\widetilde{X}, \mathbb{Z})$$

and $IH_*(X, \mathbb{Z})$ will satisfy Poincare duality. However, to build up the machinery for this, the rest of the talk seemingly will look like it has almost nothing to do with what I just said. So let's all take a deep breath and...

2 Local Systems

Definition 2.1. *Let X be a topological space and let $\Bbbk$ be a field. The constant sheaf $\Bbbk_X$ is defined as*

$$\Bbbk_X(U) = \{f : U \to \Bbbk \mid f \text{ is continuous and } \Bbbk \text{ has the discrete topology}\}$$

Remark. Equivalently, $\Bbbk_X$ is the sheaf whose sections are locally constant functions $f : U \to \Bbbk$ and also is equivalent to the sheafification of the constant presheaf which assigns $\Bbbk$ to every open set.

Remark. When U is connected, $\Bbbk_X(U) = \Bbbk$.

Definition 2.2. *A $\Bbbk-$local system on a topological space X is a sheaf $\mathcal{L} \in mod(\Bbbk_X)$ s.t. there exists a covering of X by $\{U_i\}$ s.t. $\mathcal{L}|_{U_i} = \underline{\Bbbk^{n_i}}$ where $\underline{\Bbbk^{n_i}}$ is the constant sheaf associated to the vector space $\Bbbk^{n_i}$. In other words, a local system is the same thing as a locally constant sheaf.*

Remark. If X is connected, then all the n_i are the same.

Example 1. $\Bbbk_X$ is an $\Bbbk-$local system.

Example 2. Let D be an open connected subset of $\mathbb{C}$. Then the sheaf $\mathscr{F}$ of solutions to LODE, namely

$$\mathscr{F}(U) = \left\{ f : U \to \mathbb{C} \mid f^{(n)} + a_1(z) f^{(n-1)} + \ldots + a_n(z) = 0 \right\}$$

where $a_i(z)$ are holomorphic forms a $\mathbb{C}-$local system. Existence and uniqueness of solutions of ODE on simply connected regions means that by choosing a disc $D(z)$ around each point $z \in D$, we see that the initial conditions $f^{(k)} = y_k$ give an isomorphism

$$\mathscr{F}|_{D(z)} \cong \underline{\mathbb{C}^n}$$

The stalk of the constant sheaf $\underline{\Bbbk^{n_i}}$ is just $\Bbbk^{n_i}$ as "locally constant functions" are eventually constant. Thus $\Bbbk-$local systems are sort of like vector bundles or locally free sheaves in that the stalks will be vector spaces. But they aren't the same thing because local systems are "discrete." However, local systems are essentially vector bundles with a flat connection which is an incarnation of the Riemann-Hilbert correspondence.

2.1 Monodromy

Lemma 2.3. *Any $\Bbbk-$local system $\mathcal{L}$ on a locally connected and simply connected space is a constant sheaf $\underline{V}$ for some $\Bbbk-$module V.*

I won't prove this since I did it last semester in another seminar.

Remark. As S^2 is simply connected, locally path connected, any $\mathbb{C}$ local system is constant(trivial) on S^2. However, the same isn't true for vector bundles. Complex line bundles on S^2 up to isomorphism are in bijection with $\pi_1(\mathrm{GL}_1(\mathbb{C})) = \pi_1(\mathbb{C}^\times) = \mathbb{Z}$ by the clutching construction. Or you can classify them by their first Chern class in $H^2(X, S^2) = \mathbb{Z}$. Or if you like algebraic geometry, then $S^2 = \mathbb{P}^1(\mathbb{C})$ and line bundles on $\mathbb{P}^1$ are exactly given by $\mathcal{O}_X(n)$ for $n \in \mathbb{Z}$. Thus local systems are "simpler" in the sense that you only need the first homotopy group to vanish to be trivial while vector bundles need all homotopy groups to vanish.

Theorem 1
Assume X has a universal cover. Then the following categories are equivalent.

(i) *$\Bbbk-$local systems on X.*

(ii) *Representations $\rho : \pi_1(X,x_0) \to \mathrm{GL}_\Bbbk(V)$, V is an $\Bbbk-$vector space.*

Proof. We describe how a local system $\mathcal{L}$ gives maps between fibers/stalks. Consider a path $\gamma : I \to X$ between x_0 and x_1, then note that $\gamma^{-1}L$ will be a local system on $[0, 1]$ as given a trivializing cover $\{U_i\}$ for $\mathcal{L}$, $\gamma^{-1}(U_i)$ will give us a trivializing cover for $\gamma^{-1}\mathcal{L}$. Now as $[0, 1]$ is simply connected by Lemma 2.3, we see that $\gamma^{-1}\mathcal{L} = \underline{V}$ will be a constant sheaf. As $[0, 1]$ is connected, $\underline{V}([0, 1]) = V$. It follows that the natural map $\gamma^{-1}\mathcal{L}([0, 1]) = \underline{V}([0, 1]) \to \underline{V}_x = (\gamma^{-1}\mathcal{L})_x$ sending m to the germ $(m, [0, 1])$ is an isomorphism for any $x \in [0, 1]$ (Notice that this phenomenom that global sections is isomorphic to the stalk doesn't happen for trivial $\mathcal{O}_X$ modules). Applying this to $x = 0, 1$ we obtain a chain of <u>explicit</u> isomorphisms $\gamma_* : L_{\gamma(0)} \to L_{\gamma(1)}$ called the <u>monodromy map</u>

$$\mathcal{L}_{\gamma(0)} \cong (\gamma^{-1}\mathcal{L})_0 \cong \gamma^{-1}\mathcal{L}(I) \cong (\gamma^{-1}\mathcal{L})_1 \cong \mathcal{L}_{\gamma(1)}$$

where the first isomorphism is the isomorphism of the stalk of the pullback sheaf, i.e. we send $(s, U) \mapsto ((s, U), \gamma^{-1}(U))$ and similarily with the last isomorphism. It turns out that γ_* satisfy a bunch of nice properties, namely

Lemma 2.4. *Suppose $\mathcal{L} \in mod(\Bbbk_X)$. Then the monodromy map γ_* is*

1. *$\Bbbk-$linear: $\gamma_*(v + aw) = \gamma_*(v) = a\gamma_*(w)$.*
2. *homotopy invariant: if $\gamma \sim \gamma'$, then $\gamma_* = \gamma'_*$.*
3. *compatible with composition of paths: $\gamma'_*(\gamma_*(x)) = (\gamma'.\gamma)_*(x)$*

This shows that the assignment of each point of x to it's stalk gives us a functor from the fundamental groupoid $\Pi(X)$ to $\Bbbk-$modules. By considering loops, composition of paths and homotopy invariance gives us a group homomorphism $\pi_1(X,x_0) \to Aut(\mathcal{L}_{x_0}) := Aut(V)$ and by $\Bbbk-$linearity it will land in $\mathrm{GL}_{\Bbbk}(V)$. This map

$$\pi_1(X,x_0) \to \mathrm{GL}_{\Bbbk}(V)$$

is called the **monodromy representation** of $\mathcal{L}$. To go back consider the space $(\widetilde{X} \times V)/\pi_1(X,x_0)$ over X where $\pi_1(X,x_0)$ acts on $\widetilde{X}$ by monodromy and V has the discrete topology. Take it's sheaf of sections to recover $\mathcal{L}$.

Lemma 2.5. *The monodromy map on a constant sheaf* $\underline{A}$ *is trivial.*

Remark. We can now give a geometric picture of the monodromy map. By Lemma 2.5 we can think of monodromy on an element $a \in \mathcal{L}_x$ as a composition of piecewise constant maps corresponding to where $\mathcal{L}$ is constant as we go along γ. [draw picture of loop and a cover of the loop] On the overlaps a might change when identifying the stalks and this is what gives rise to the action.

You can now go home and generate a lot of examples of local systems by taking representations of fundamental groups.

3 Verdier Duality

Definition 3.1 (Direct Image with Compact Support)**.** *Let* $f: X \to Y$ *be a continuous map. Then* $f_!$ *is defined to be the functor*

$$f_!\mathcal{F}(V) = \left\{s \in \Gamma(f^{-1}(V), \mathcal{F}) \,|\, f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to V \textit{ is proper}\right\}$$

where proper is in the topological sense, i.e. inverse image of compact is compact.

Remark. It follows that when f is proper, $f_! = f_*$. Moreover if f is pushforward to a point, $a_X : X \to \{pt\}$, we recover cohomology with compact support, i.e. for $\mathcal{F} \in \mathrm{Sh}(X)$ we have that $(a_X)_!(\mathcal{F}) = \Gamma_c(X, \mathcal{F})$ where

$$\Gamma_c(X,\mathcal{F}) = \{s \in \Gamma(X,\mathcal{F}) \,|\, \mathrm{supp}(s) \text{ is compact } \}$$

When f is proper $f_! = f_*$ has a left adjoint given by f^{-1}. Can $f_!$ have a right adjoint? In general no, as when f is proper this would imply that $f_!$ is exact, as $f_! = f_*$ being a right adjoint implies it preserves colimits and thus cokernels and $f_!$ being a left adjoint means its preserves limits and thus kernels. However $f_!$ is clearly not exact for all proper maps. For example consider $f: X \to pt$ where X is compact. If $f_!$ is exact then there are no higher derived functors, which in this case would mean there's no higher cohomology with compact support for any sheaf $\mathcal{F}$ on compact spaces X which is just cohomology since X is compact which clearly isn't true.

However if we move to the derived category we will obtain our desired adjunction.

Theorem 2 (Global Verdier Duality)
Let $f: X \to Y$ *be a continuous map. Then there is an additive triangulated functor* $f^! : D^+(Y) \to D^+(X)$*, called exceptional inverse image such that we have an adjunction*

$$\mathrm{Hom}_{D^+(Y)}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{D^+(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$$

Proof. Use some adjoint functor theorem.

Remark. When $j : U \hookrightarrow X$ is an open embedding, $j^! = j^{-1}$ (So not only does $j^!$ exist at the level of abelian categories we don't need to derive it because j^{-1} is exact.)

Definition 3.2. *Let $a_X : X \to pt$ be the usual projection to a point. Then the dualizing complex is defined as*

$$\omega_{X/A} = a_X^!(\underline{A}_{pt})$$

Although we haven't explicitly constructed the functor $a_X^!$ we actually can obtain a very good description of the cohomology sheaves of the complex $\omega_{X/A}$ where $A = \Bbbk$ is a field.

Lemma 3.3. *Suppose $A = \Bbbk$ is a field and let $\omega_X = \omega_{X/\Bbbk}$. Then for every integer i the cohomology sheaf $\mathcal{H}^i\omega_X$ is the sheafification of the presheaf*

$$U \mapsto H_c^{-i}(U, \Bbbk)^\vee$$

Proof. We will evaluate $\mathrm{Hom}_{D^+(X)}(\underline{\Bbbk}_U, \omega_{U/\Bbbk}[i]) = \mathrm{Hom}_{D^+(X)}(\underline{\Bbbk}_U, a_U^!(\underline{\Bbbk}_{pt})[i])$ in two different ways.

$$\mathrm{Hom}_{D^+(U)}(\underline{\Bbbk}_U, a_U^!(\underline{\Bbbk}_{pt})[i]) = H^i(R\mathrm{Hom}^\bullet_{\underline{\Bbbk}_U}(\underline{\Bbbk}_U, a_U^!(\underline{\Bbbk}_{pt})) = \mathbb{H}^i(U, a_U^!(\underline{\Bbbk}_{pt}))$$

where the last equality comes from resolving $a_U^!(\underline{\Bbbk}_{pt})$ and noting that $\mathrm{Hom}_{\underline{\Bbbk}_U}(\underline{\Bbbk}_U, \mathcal{I}) = \mathcal{I}(U)$ so we are just computing hypercohomology. On the other hand by the adjunction we also have that

$$\begin{aligned}\mathrm{Hom}_{D^+(U)}(\underline{\Bbbk}_U, a_U^!(\underline{\Bbbk}_{pt})[i]) &= \mathrm{Hom}_{D^+(U)}(\underline{\Bbbk}_U[-i], a_U^!(\underline{\Bbbk}_{pt})) = \mathrm{Hom}_{D^+(pt)}((a_U)_!(\underline{\Bbbk}_U[-i]), \underline{\Bbbk}_{pt}) \\ &= H^0(R\mathrm{Hom}^\bullet_{\underline{\Bbbk}_{pt}[-1]}(\,(a_U)_!(\underline{\Bbbk}_U[-i]),\, \underline{\Bbbk}_{pt}))\end{aligned}$$

By definition, to compute $(a_U)_!(\Bbbk_U)$ we pick an injective resolution of $\Bbbk_U$, $\mathcal{I}^\bullet \to \Bbbk_U$ and apply $(a_U)_!$ to each term. But since we end up in $C^*(\underline{\Bbbk}_{pt})$, chain complexes of sheaves over a point, this is the same as chain complexes of $\Bbbk$ vector spaces by sending each sheaf $(a_U)_!\mathcal{I}^k$ to it's global sections. By definition this will yield $\Gamma_c(U, \mathcal{I}^k)$. Moreover as every $\Bbbk$ module is free we don't need to derive $\mathrm{Hom}^\bullet_{\Bbbk}$ and so we have the equality

$$\begin{aligned}\mathrm{Hom}_{D^+(U)}(\underline{\Bbbk}_U, a_U^!(\underline{\Bbbk}_{pt})[i]) &= H^{-i}(\mathrm{Hom}^\bullet_{\Bbbk}((0 \to \Gamma_c(U, \mathcal{I}^0) \to \Gamma_c(U, \mathcal{I}^1) \to \ldots), \Bbbk)) \\ &= H^{-i}(\ldots \to \Gamma_c(U, \mathcal{I}^1)^\vee \to \Gamma_c(U, \mathcal{I}^0)^\vee \to 0) = H_c^{-i}(U, \Bbbk)^\vee\end{aligned}$$

Finally $\mathcal{H}^i\omega_X$ is isomorphic to the corresponding cohomology sheaf of the injective resolution we took of $a_U^!(\underline{\Bbbk}_U)$ to obtain hypercohomology. Quotient sheaves are sheafification of the naive quotient and thus $\mathcal{H}^i\omega_X$ is the sheafification of the naive presheaf for hypercohomology

$$U \mapsto \mathbb{H}^i(U, a_U^!(\underline{\Bbbk}_{pt}))$$

Our calculations above show that $\mathbb{H}^i(U, a_U^!(\underline{\Bbbk}_{pt})) = H_c^{-i}(U, \Bbbk)^\vee$ and thus the result.

So now we have a good understanding of the cohomology sheaves of $\omega_{X/\Bbbk}$, and in some cases this will actually determine $\omega_{X/\Bbbk}$. Namely first consider

Lemma 3.4. *Let A be any ring. Then we have that*

$$H_c^i(\mathbb{R}^n, A) \cong \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Proof. This is more or less results from the fact that the one point compactification $\mathbb{R}^n \cup \{\infty\} = S^n$.

Corollary 3.5. *Let $\Bbbk$ be a field and let X be an $n-$dimensional manifold. Then the dualizing complex $\omega_{X/\Bbbk} \in D^+(X)$ on X is isomorphic to $\mathcal{H}^{-n}\omega_X$ so it's just a sheaf.*

Proof. Because X is a manifold, the presheaf assigning U it's compactly supported cohomology

$$U \mapsto H_c^{-i}(U, \Bbbk)^\vee$$

is zero for all open ball U by the above lemma except when $i = -n$. As open balls generate the topology for X it follows that all cohomology sheaves except $i = -n$ are zero. Whenever a chain complex has cohomology only supported in one degree, it turns out that the chain complex will be quasi-isomorphic to that single cohomology group as follows. Resolve(find a quasi-isomorphism) $\omega_{X/\Bbbk} \to \mathcal{I}^\bullet$ using a complex of injectives. Because $\mathcal{I}^{-n}$ is injective, we can split off the submodule $\ker d_{-n}$ inside so that $\mathcal{I}^{-n} = \ker d_{-n} \oplus \mathcal{J}$. Now we have a morphism of chain complexes

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathcal{I}^{-n-1} & \longrightarrow & \ker d_{-n} \oplus \mathcal{J} & \xrightarrow{d_{-n}} & \mathcal{I}^{-n+1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \ker d_{-n}/\mathrm{im} d_{-n-1} & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

This morphism will be a quasi-iso by our previous computation and thus we have that $\mathcal{I}^\bullet$ is quasi-isomorphic to $\mathcal{H}^{-n}\omega_X$ and therefore so is $\omega_{X/\Bbbk}$. By definition this means that $\mathcal{H}^{-n}\omega_X$ is isomorphic to $\omega_{X/\Bbbk}$ in the derived category as desired.

Definition 3.6. *Let X be an $n-$dimensional topological manifold. Then the orientation sheaf $\mathcal{L}_{or}$ is the sheaf associated to the presheaf $U \mapsto H_c^n(U, \Bbbk)^\vee$. (This actually turns out to already be a sheaf)*

Remark. We see that the above corollary shows that when X is an $n-$dimensional manifold, then $\omega_{X/\Bbbk} \cong \mathcal{L}_{or}[n]$. Moreover contrast this situation with Serre duality in algebraic geometry where given a Cohen-Macaualy scheme we have that the dualizing sheaf ω_X

- If $f : X \to Y$ is a finite morphism between locally Noetherian schemes, then $\omega_f = f^!\mathcal{O}_Y$.
- If X is smooth then $\omega_X \cong \wedge^n \Omega_X$.

In fact if we use Grothendieck duality the analogy becomes even more clear.

Recall in the proof of Lemma 3.3 we were able to relate (hyper)cohomology of a sheaf with compactly supported cohomology by evaluating $\mathrm{Hom}_{D^+(X)}(\underline{\Bbbk}_U, \omega_{U/\Bbbk}[i])$ two different ways. To generalize this phenomenom, we introduce

Definition 3.7 (Verdier Dual)**.** *Let X be a topological space and let $\mathcal{F}^\bullet \in D^b(X)$ define $\mathbb{D}_X(\mathcal{F}^\bullet) \in D^b(X)$*

$$\mathbb{D}_X(\mathcal{F}^\bullet) = R\mathscr{H}om^\bullet(\mathcal{F}^\bullet, \omega_X)$$

This will be a contravariant functor on $D^b(X)$.

Remark. In the special case where X is a topological manifold, we know from Corollary 3.5, that $\omega_X \cong \mathcal{L}_{or}[n]$ is just a sheaf. As a result, a little more work will then show that if $\mathcal{F}^\bullet = \mathcal{F}$ is just a sheaf, we have that

$$\mathbb{D}(\mathcal{F}^\bullet) = \mathscr{H}om(\mathcal{F}, \mathcal{L}_{or})[n]$$

In the special case when $\mathcal{F}$ is also a local system $\mathcal{L}$ we have the additional isomorphism that

$$\mathbb{D}(\mathcal{L}) = \mathscr{H}om(\mathcal{L}, \underline{A}_X) \otimes \mathcal{L}_{or}[n] = \mathcal{L}^\vee \otimes \mathcal{L}_{or}[n]$$

In terms of representation if (M, ρ) is the monodromy representation of $\mathcal{L}$, then the monodromy representation $\rho^\vee$ corresponding to $\mathcal{L}^\vee$ on $M^\vee$ is given by

$$\rho^\vee([\gamma])(\varphi)(m) = \varphi(\rho([\gamma])^{-1}m) \quad \forall \varphi \in M^\vee, m \in M$$

4 Constructible Sheaves

X for the most part will be a complex analytic space from now on so we have to distinguish between real and complex dimensions at times. This means varieties do not have the Zariski topology (aka we aren't doing algebraic geometry) because local systems on the Zariski topology is bad. As usual to fix this we would need to consider $\ell-$adic sheaves in the etale topology but that's another talk.

Example 3. The pushforward of a local system is not necessarily a local system. Consider the pushfoward of a nontrivial local system $\mathcal{L}$ under the inclusion $j : \mathbb{C}^\times \to \mathbb{C}$. If $j_*\mathcal{L}$ is a local system it is necessarily the trivial one since $\pi_1(\mathbb{C}) = 1$. However if we compute the monodromy map for $j_*\mathcal{L}$ then it will just be the monodromy map for $\mathcal{L}$ on $\mathbb{C}^\times$ because locally around the loop we will always be in $\mathbb{C}^\times$ and so on sections and on stalks the computations are all on $\mathbb{C}^\times$. Thus the monodromy map is nontrivial but this is impossible since $j_*\mathcal{L}$ is supposed to be the trivial local system on $\mathbb{C}$.

Unlike local systems, constructible sheaves will be preserved under pushforward.

Definition 4.1. *Let X be a topological space. A stratification of X is a partially ordered set $(\Lambda, \leq)$ and a collection of locally closed subsets $\{X_\lambda\}_{\lambda\in\Lambda}$ such that*

1. $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$ *and* $\overline{X_\lambda} = \bigsqcup_{\mu\leq\lambda} X_\mu$.
2. *Each X_λ is a smooth connected complex manifold.*

Example 4. Let G be a connected reductive group. Then we have the Bruhat decomposition for G given by

$$G = \bigsqcup_{w\in W} BwB$$

Therefore the flag variety G/B is stratified by the B orbits on G, i.e.

$$G/B = \bigsqcup_{w\in W} BwB/B$$

$BwB/B \cong \mathbb{C}^{\ell(w)}$ are called Schubert cells and their closure (in either Zariski or complex analytic) satisfy

$$\overline{BwB/B} = \bigsqcup_{x\leq w} BxB/B$$

where $\leq$ is the Bruhat order on W. $\overline{BwB/B}$ are called Schubert varieties and tend to be singular.

Definition 4.2. *A sheaf $\mathcal{F} \in \text{mod}(\underline{A}_X)$ is constructible if there exists a stratification $\bigsqcup_{\lambda\in\Lambda} X_\lambda$ such that $\mathcal{F}|_{X_\lambda}$ is a local system of <u>finite</u> rank for all $\lambda \in \Lambda$.*

Definition 4.3. *A complex* $\mathcal{F}^\bullet \in D^b(X, A)$ *is constructible if all its* *cohomology sheaves*[1] $\mathcal{H}^m\mathcal{F}^\bullet$ *are constructible for some stratification* Λ*. Let*

$$D^b_c(X, A) = \Big\{\textit{full triangulated subcategory of } D^b(X, A) \textit{ consisting of constructible complexes}\Big\}$$

Theorem 4.4 (6 functors formalism)**.** $D^b_c(X)$ *is closed under the six operations*

$$Rf_*, Rf_!, f^{-1}, f^!, R\mathscr{H}om, \otimes^L$$

Corollary 4.5. *The dualizing sheaf* ω_X *is in* $D^b_c(X)$*. More generally* $\mathbb{D}$ *descends to a functor*

$$\mathbb{D} : D^b_c(X, A) \to D^b_c(X, A)$$

[1]If we were to ask that each term in $\mathcal{F}^\bullet$ is constructible, this would not be well defined in the derived category; a different representative might actually have different sheaves, as we only know that the cohomology sheaves are the same.