Category $\mathcal{O}$: Properties

All of this information comes from *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$* by James E. Humphreys.

1 Preliminaries/Notation

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Let $\Phi \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, and for any $\alpha \in \Phi$ define

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

Fixing a simple system $\Delta \subset \Phi$ gives a Cartan decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus n$, where $\mathfrak{n} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \oplus_{\alpha<0}\mathfrak{g}_\alpha$.

For a $U(\mathfrak{g})$-module M we define

$$M_\lambda = \{v \in M | h \cdot v = \lambda(h)v \; \forall h \in \mathfrak{h}\}$$

2 What is Category $\mathcal{O}$?

Definition 1. *The **BGG category** $\mathcal{O}$ is the full subcategory of* $\operatorname{Mod} U(\mathfrak{g})$ *whose objects are modules such that:*

$\mathcal{O}$1. M is a finitely generated $U(\mathfrak{g})$-module

$\mathcal{O}$2. M is $\mathfrak{h}$-semisimple (i.e. M is a weight-module: $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$)

$\mathcal{O}$3. M is locally $\mathfrak{n}$-finite: for each $v \in M$, the subspace $U(\mathfrak{n}) \cdot v \subseteq M$ is finite dimensional.

An immediate consequence is that all finite dimensional modules are in category $\mathcal{O}$. These axioms also imply that:

$\mathcal{O}$4. *All weight spaces of M are finite dimensional*

$\mathcal{O}$5. *The set of all weights of M is contained in a finite union of sets of the form $\lambda - \Gamma$, where $\lambda \in \mathfrak{h}^*$ and $\Gamma \subset \Lambda_r$ (where Λ_r is the root lattice) is the semigroup generated by Φ^+.*

2.1 Basic Properties

Theorem 1. Category $\mathcal{O}$ satisfies:

a) $\mathcal{O}$ is a noetherian category (each $M \in \mathcal{O}$ is a noetherian $U(\mathfrak{g})$-module).

b) $\mathcal{O}$ is closed under submodules, quotients, and finite direct sums.

c) $\mathcal{O}$ is an abelian category.

d) If $M \in \mathcal{O}$ and $L \in \operatorname{Mod} U(\mathfrak{g})$ is finite dimensional, then $L \otimes M \in \mathcal{O}$ (so $M \mapsto L \otimes M$ defines an exact functor $\mathcal{O} \to \mathcal{O}$)

e) If $M \in \mathcal{O}$ then M is $Z(\mathfrak{g})$-finite (for each $v \in M$, the span of $\{z \cdot v | z \in Z(\mathfrak{g})\}$ is finite dimensional).

f) If $M \in \mathcal{O}$ then M is finitely generated as a $U(\mathfrak{n}^-)$-module.

We will prove Part (d). Let $M \in \mathcal{O}$ and let $L \in \operatorname{Mod} U(\mathfrak{g})$ be finite dimensional. We need to check the axioms $\mathcal{O}1 - \mathcal{O}3$.

$\mathcal{O}1$. Let $v_1, \ldots, v_n$ be a basis of L and $w_1, \ldots, w_p$ a generating set for M. Let N be the submodule generated by the elements $v_i \otimes w_j$. Clearly, $N \subset L \otimes M$. For the reverse containment, let $v \in L$. Then for any j, $v \otimes w_j \in N$. Let $x \in \mathfrak{g}$. Then

$$x \cdot (v \otimes w_j) = x \cdot v \otimes w_j + v \otimes x \cdot w_j \in N$$

Since L is itself a module, $x \cdot v \in L$, and so $v \otimes x \cdot w_j \in N$. Iteration (since L is finite dimensional) shows that $v \otimes u \cdot w_j \in N$ for all PBW monomials $u \in U(\mathfrak{g})$, so that $L \otimes M \subset N$. Thus $L \otimes M = N$, so that $L \otimes M$ is a finitely generated $U(\mathfrak{g})$-module. $\square$

$\mathcal{O}2$. M is a weight module by $\mathcal{O}2$. Since all finite dimensional modules are weight modules (Section 0.8) L is too. Therefore $L \otimes M$ is a weight module. $\square$

$\mathcal{O}3$. By assumption, M is locally $\mathfrak{n}$-finite. Being finite dimensional, any subspace of L is also finite dimensional, so for each $v \in L \otimes M$, $U(\mathfrak{n}) \cdot v \subset L \otimes M$ is also finite dimensional. $\square$

2.2 Highest Weight Modules

Definition 2. *Let $M \in U(\mathfrak{g}) - \text{Mod}$. Then $v^+ \in M \setminus \{0\}$ is a* ***maximal vector*** *of weight $\lambda \in \mathfrak{h}^*$ if $v^+ \in M_\lambda$ and $\mathfrak{n} \cdot v^+ = 0$.*

Definition 3. *$M \in U(\mathfrak{g}) - \text{Mod}$ is a* ***highest weight module*** *of weight λ is there is a maximal vector $v^+ \in M_\lambda$ such that $M = U(\mathfrak{g}) \cdot v^+$.*

Theorem 2a. Highest weight modules are in category $\mathcal{O}$. Let M be a highest weight module of weight λ.

$\mathcal{O}1$. As a $U(\mathfrak{g})$-module, M is generated by v^+ for some $v^+ \in M_\lambda$. $\square$

$\mathcal{O}2$. We want to show that M is $\mathfrak{h}$-semisimple, i.e. that $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$. We know that M is spanned by elements of the form $y_1^{i_1} \dots y_m^{i_m} \cdot v^+$, where $i_j \in \mathbb{Z}^+$ for all j. An element of this form has weight $\lambda - \sum i_j \alpha_j$ (where y_j lies in the root space $\mathfrak{g}_{\alpha_j}$). Weight vectors of distinct weights are linearly independent, so the commutation relations for $\mathfrak{h}$ and $\mathfrak{n}^-$ give us the decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, so that M is $\mathfrak{h}$-semisimple. □

$\mathcal{O}3$. Let $v \in M$. Then for any $u \in U(\mathfrak{n})$, the weight of $u \cdot v$ is at least that of v. If infinitely many $u \in U(\mathfrak{n})$ raised v to the same weight space, that weight space would be infinite dimensional (note that $\mathfrak{g}_\alpha \cdot M_\mu \subset M_{\mu+\alpha}$). This is impossible, so M is locally $\mathfrak{n}$-finite.

□

Theorem 2b. Highest weight modules are indecomposable.

Proof. Each proper submodule of M is a weight module. Since $M = U(\mathfrak{g}) \cdot v^+ = U(\mathfrak{g}) \cdot M_\lambda$, any proper submodule of M cannot have λ as a weight. Therefore the sum of all proper submodules is itself proper, so M has a unique maximal submodule, hence is indecomposable.

□

Corollary - from Page 17 . Let $M \neq 0 \in \mathcal{O}$. Then M has a finite filtration $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ such that each M_i / M_{i-1} is a (nonzero) highest weight module.

Proof. By $\mathcal{O}1$, M can be generated by finitely many weight vectors $\{v_1, \dots, v_n\}$. So, using $\mathcal{O}3$ (i.e. that for any $v \in M$, $U(\mathfrak{n}) \cdot v$ is finite dimensional) The $\mathfrak{n}$-submodule V generated by $\{v_1, \dots, v_n\}$ is also finite dimensional. If $\dim V = 1$, then M is already a highest weight module. Otherwise, we can induct on $\dim V$. Take $v \neq 0 \in V$ a weight vector of weight λ, such that λ is maximal among all weights of V (we can do this since V was generated by finitely many weight vectors). $M_1 := U(\mathfrak{g}) \cdot v$ is a submodule of M, and so lies in $\mathcal{O}$. Furthermore, $\overline{M} := M/M_1$ also lies in $\mathcal{O}$ and is generated by the image $\overline{V}$ of V under the same quotient. $\dim \overline{V} < \dim V$, so we can apply the induction hypothesis to $\overline{M}$ to obtain a chain of highest weight submodules which we can lift back to M. □

3 The Length of Category $\mathcal{O}$

Our next task is to prove that the category is of finite length. To do this, we will begin by looking closer at the action of $Z(\mathfrak{g})$. We know that any $M \in \mathcal{O}$ is locally finite as a $Z(\mathfrak{g})$-module (for any $v \in M$, the span of $\{z \cdot v | z \in Z(\mathfrak{g})\}$ is finite dimensional). If M is a highest weight module $M = U(\mathfrak{g}) \cdot v^+$, of weight λ, then for any $z \in Z(\mathfrak{g})$ and $h \in \mathfrak{h}$ we have that

$$\begin{aligned} h \cdot (z \cdot v^+) &= z \cdot (h \cdot v^+) && (z \in Z(\mathfrak{g})) \\ &= z \cdot (\lambda(h) v^+) && (v^+ \in M_\lambda) \\ &= \lambda(h) z \cdot v^+ \end{aligned}$$

So, $z \cdot v^+ = \chi_\lambda(z)v^+$ for some $\chi_\lambda(z) \in \mathbb{C}$ (since $\dim M_\lambda = 1$). Since all elements of M are of the form $u \cdot v^+$ for some $u \in U(\mathfrak{n}^-)$ we also know that

$$\begin{aligned} z \cdot (u \cdot v^+) &= u \cdot (z \cdot v^+) \\ &= \chi_\lambda(z) u \cdot v^+ \end{aligned}$$

Thus the action of the center completely determines the action on a highest weight module. For any fixed λ, we will call $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ the *central character associated with* λ, and more generally define:

Definition 4. *A* ***central character*** *is an algebra homomorphism* $Z(\mathfrak{g}) \to \mathbb{C}$.

We will see shortly that any central character can be written as χ_λ for some weight λ.

Now, let's look a little closer at χ_λ for some fixed λ. By the triangular decomposition, for any $z \in Z(\mathfrak{g})$ we can write as a linear combination of PBW monomials, and $z \cdot v^+$ will depend only on the monomials with factors in $\mathfrak{h}$. So, letting

$$pr : \begin{cases} U(\mathfrak{g}) \to U(\mathfrak{h}) \\ x_i, y_i \mapsto 0 \\ h_i \mapsto h_i \end{cases}$$

we see that $\chi_\lambda(z) = \lambda(pr(z))$ (where $\lambda \in \mathfrak{h}^*$ is extended to an algebra homomorphism $U(\mathfrak{h}) \to \mathbb{C}$).

Since $\bigcap_{\lambda \in \mathfrak{h}^*} \ker \lambda = 0$, the **Harish-Chandra homomorphism** defined by $\xi = pr|_{Z(\mathfrak{g})}$ is an algebra homomorphism $\xi : Z(\mathfrak{g}) \to U(\mathfrak{h})$.

Definition 5. *Let* $w \in W$ *(the Weyl group) and* $\lambda \in \mathfrak{h}^*$*. We define the* ***dot action****, a shifted action of* W*, by*

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

(where $\rho = \frac{1}{2} \sum_{\lambda \in \Phi^+} \lambda$).

Definition 6. *The* ***linkage class*** *of* λ *is the orbit* $\{w \cdot \lambda | w \in W\}$ *of* λ *under the dot action. We say that two elements of the same linkage class are* ***linked****.*

Definition 7. *A weight* $\lambda \in \mathfrak{h}^*$ *is called a* ***regular weight*** *(or* ***dot-regular****) if* $|W \cdot \lambda| = |W|$.

Definition 8. *A* ***singular weight*** *is a weight which is not regular.*

For a given $\lambda \in \mathfrak{h}^*$, the linkage class of λ has a unique element in $\overline{C} - \rho$ (where $C = \{\mu \in E | \langle \mu, \alpha^\vee \rangle > 0 \; \forall \alpha \in \Delta\}$ is the Weyl chamber.

We define the **twisted Harish-Chandra homomorphism** as

$$\psi : \begin{cases} Z(\mathfrak{g}) \to U(\mathfrak{h}) = S(\mathfrak{h}) \\ z \mapsto \tau_\rho(\xi(z)) \end{cases}$$

Theorem 3. [Harish-Chandra] Let $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h}) = P(\mathfrak{h}^*)$ be the twisted Harish-Chandra homomorphism.

a) ψ is an isomorphism onto $S(\mathfrak{h})^W \subset S(\mathfrak{h})$.

b) $\forall \lambda, \mu \in \mathfrak{h}^*$, we have $\chi_\lambda = \chi_\mu \iff \exists w \in W$ such that $\mu = s \cdot \lambda$ (i.e. λ and μ are W-linked).

c) Every central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$.

Outline of a proof for (a): We begin by noting that $\psi(Z(\mathfrak{g})) \subset S(\mathfrak{h})^W$. We consider the algebra of polynomial functions on $\mathfrak{g}$ considered as a vector space, $P(\mathfrak{g}) \cong S(\mathfrak{g}^*)$. Then the restriction $\theta : P(\mathfrak{g}) \to P(\mathfrak{h})$ is an algebra homomorphism. The adjoint group $G \subset Aut\mathfrak{g}$ generated by $\exp \operatorname{ad} x$ for nilpotent x is a Lie group which acts naturally on $P(\mathfrak{g})$. Similarly, W acts on $P(\mathfrak{h})$. Chevalley proved that $P(\mathfrak{g})^G \cong P(\mathfrak{h})^W$ via the restriction map θ. Identifying $P(\mathfrak{a})$ with $S(\mathfrak{a})$ for $\mathfrak{a} = \mathfrak{g}, \mathfrak{h}$, we obtain enough information via comparison to ξ so see that ψ is bijective, hence an isomorphism. □

b). We first assume that $\lambda \in \Lambda$ and that μ, λ are in the same linkage class. Let $\alpha \in \Delta$. Then $n := \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$. If $n \geq 0$, then $M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda)$, so that $\chi_\lambda = \chi_{s_\alpha \cdot \lambda} = \chi_\mu$. If $n = -1$, then $s_\alpha \cdot \lambda = \lambda$ and we are done. If $n < -1$, then letting $\mu = s_\alpha \cdot \lambda$, we obtain

$$\langle \mu, \alpha^\vee \rangle = -n - 2 \geq 0$$

so by the first case, $\chi_\lambda = \chi_\mu$. Since W is generated by simple reflections, and linkage is a transitive relation, by induction on $\ell(w)$ we obtain that

$$\mu = w \cdot \lambda \implies \chi_\lambda = \chi_\mu$$

(i.e. if λ, μ lie in the same linkage class, then they induce the same central characters.

Now, We identify $\mathfrak{h}^*$ with $\mathbb{A}^\ell$, the affine space over $\mathbb{C}$. We then can identify $U(\mathfrak{h}) = S(\mathfrak{h}) = P(\mathfrak{h}^*)$ with the algebra of polynomial functions acting on $\mathbb{A}^\ell$, and the integer lattice λ with $\mathbb{Z}^\ell$. Since $\mathbb{Z}^\ell$ is Zariski dense in $\mathbb{A}^\ell$, by the above result we know that $\chi_\lambda = \chi_{w \cdot \lambda}$ for any $\lambda \in \mathfrak{h}^*$.

Now, assume that λ and μ lie in disjoint linkage classes. Let $f \in P(\mathfrak{h}^*)$ be a polynomial such that $f|_{W(\lambda+\rho)} = 1$ and $f|_{W(\mu+\rho)} = 0$. Then, define

$$g := \frac{1}{|W|} \sum_{w \in W} wf$$

g is W-invariant and agrees with f on the specified W-orbits. Using part (a), we can take any $z \in \psi^{-1}(g) \subset Z(\mathfrak{g})$. Then

$$\chi_\lambda(z) = (\lambda + \rho)\psi(z) = g(\lambda) = 1,$$

but

$$\chi_\mu(z) = (\mu + \rho)\psi(z) = g(\mu) = 0$$

which means $\chi_\lambda \neq \chi_\mu$.

Therefore, $\chi_\lambda = \chi_\mu$ if and only if λ and μ lie in the same linkage class. □

c). We want to show that every central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$. Let χ be an arbitrary central character. Via ψ we have that χ corresponds to a homomorphism $\varphi : S(\mathfrak{h})^W \to \mathbb{C}$. Since the Weyl group is finite, $S(\mathfrak{h})$ is an integral extension of $S(\mathfrak{h})^W$. So (via the Going Up Theorem), φ extends to a homomorphism $\tilde{\varphi} : S(\mathfrak{h}) \to \mathbb{C}$. Now, since $S(\mathfrak{h}) = P(\mathfrak{h}^*)$, $\exists \lambda \in \mathfrak{h}^*$ such that $\tilde{\varphi} = \mathrm{eval}_{\lambda+\rho}$. This gives us that, for any $z \in Z(\mathfrak{g})$,

$$\chi(z) = (\lambda + \rho)(\psi(z)) = \chi_\lambda(z)$$

as desired. □

Theorem 4. Category $\mathcal{O}$ is artinian.

Proof. By the Corollary from page 17, it suffices to prove that Verma modules $M(\lambda)$ are artinian. Let $V := \sum_{w \in W} M(\lambda)_{w \cdot \lambda}$. Note that $\dim V < \infty$. Let $N' \subset N$ (proper containment) be submodules of $M(\lambda$. Then $Z(\mathfrak{g})$ acts on N/N' by the character χ_λ. N/N' has a maximal weight vector of some weight $\mu \leq \lambda$, so $\chi_\mu = \chi_\lambda \implies \exists w \in W$ such that $\mu = w \cdot \lambda$. This implies that $N \cap V \neq 0$, and $\dim(N \cap V) > \dim(N' \cap V)$. Therefore any properly descending chain of submodules of $M(\lambda)$ terminates in finitely many steps, so $\mathcal{O}$ is artinian. □

So, the category $\mathcal{O}$ is both artinian and noetherian, and hence is of finite length.

4 Subcategories $\mathcal{O}_\chi$

Definition 9. *Let χ be a central character. We define*

$$M^\chi := \{v \in M | (z - \chi(z))^n \cdot v = 0 \textit{ for some } n > 0 \textit{ depending on } z\}$$

M^χ is a $U(\mathfrak{g})$-submodule of M, and for distinct χ, the corresponding M^χ's are independent.

We define the subcategory $\mathcal{O}_\chi \subset \mathcal{O}$ to be the full subcategory of $\mathcal{O}$ which objects M such that $M = M^\chi$.

Theorem 5. $\mathcal{O}$ decomposes into a direct sum

$$\mathcal{O} = \bigoplus_\lambda O_{\chi_\lambda} = \bigoplus_{\lambda \in \mathfrak{h}^* \backslash (W \cdot)} \mathcal{O}_{\chi_\lambda}$$

Proof. Since $Z(\mathfrak{g})$ and $U(\mathfrak{h})$ commute, $Z(\mathfrak{g})(M_\mu) \subset M_\mu$. So, $M_\mu = \bigoplus_\chi (M_\mu \cap M^\chi)$. Since M is generated by finitely many weight vectors, $\exists \chi_i$ such that $M = \bigoplus_{i=1}^n M^{\chi_i}$. By Harish-Chandra's theorem, there exist $\lambda_1, \ldots, \lambda_n$ such that $\chi_i = \chi_{\lambda_i}$ for each i. Since $\chi_\lambda = \chi_\mu$ for weights in the same linkage class, we can reduce this sum to just the equivalence classes under the dot action. □

Let M_1, M_2 be simple modules in the cateogry such that there exists a non-split short exact sequence $0 \to M_i \to M \to M_j \to 0$, i.e. M_1, M_2 can be extended nontrivially, then we say that they are in the same block. If for simple modules M, N there is a sequence $M = M_1, \ldots, M_n = N$ such that adjacent pairs are in the same block, we say that M and N are in the same block. For an arbitrary module M, we say that M is in a given block if all of its composition factors are.

Theorem 6. If $\lambda \in \Lambda$, then the subcategory $\mathcal{O}_{\chi_\lambda}$ is a block of $\mathcal{O}$.

Proof. We need only show that all simple modules $L(w \cdot \lambda)$ lie in the same block. First, assume that $\alpha \in \Delta$, and assume that $\mu := s_\alpha \cdot \lambda$ satisfies $\mu < \lambda$. We know that there is a nonzero homomorphism $f : M(\mu) \to N(\lambda) \subset M(\lambda)$, which induces an embedding $L(\mu) \hookrightarrow M(\lambda)/f(N)$ which has quotient $L(\lambda)$. This is a highest weight module, hence is indecomposable. So, $L(\lambda)$ and $L(\mu)$ lie in the same block. Iteration over a reduced expression for $w \in W$ gives us the result. $\square$