Geometric representation theory of quantum affine algebras

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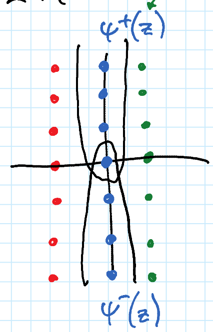
Last time: $\sum \psi_n^+ z^n$



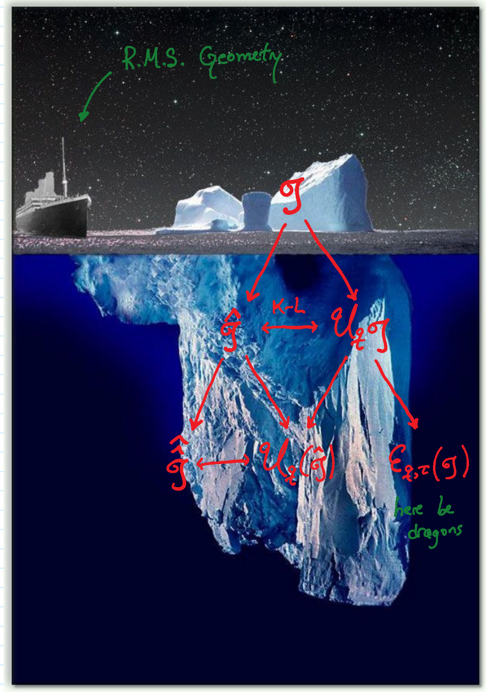
$U_q \hat{\mathfrak{g}} \leftarrow$ quantization of $U(\mathfrak{g}[t,t^{-1}])$ ← loop var.

$V_i(a) \leftarrow$ evaluation reps $(\mathfrak{g} = \mathfrak{sl})$

$(t \mapsto a$ and $V_i \in Rep\, \mathfrak{g})$

$\chi_q(V) \approx tr_V\, \psi^{\pm}(z) \leftarrow$ q-character

slight repackaging



This time: $U_q \hat{\mathfrak{g}} \leftarrow$ convolution alg. acting on K-theory of Nakajima quiver variety $\mathcal{M}_Q(\vec{w})$

$V_i(a) \leftarrow$ K-theory of $\mathcal{M}_Q(\vec{\delta}_i)$

and more!

$\chi_q(V) \leftarrow$ Euler characteristics

Nakajima quiver varieties:

usual nodes, framing nodes

$\bigcirc - \bigcirc - \square$ $\rightsquigarrow$ (take reps) $\textcircled{$v_1$} \rightleftarrows \textcircled{v_2} \rightleftarrows \boxed{w}$ $\rightsquigarrow$ (alg. sympl. red.) $\mathcal{M}_Q(\vec{v},\vec{w})$

quiver Q (with framing) $\qquad T^* Rep_Q(\vec{v},\vec{w})$

$\mathcal{M}_Q(\vec{v},\vec{w}) = T^*Rep_Q(\vec{v},\vec{w}) /\!/\!/\!/_{\xi,\theta} \prod GL(V_i)$

e.g. $\textcircled{v} - \boxed{w} \rightsquigarrow T^* Gr(v,w)$

$\textcircled{$v_1$} - \textcircled{$v_2$} - \textcircled{$v_3$} - \cdots - \textcircled{v_n} - \boxed{w} \rightsquigarrow T^* Flag(\mathbb{C}^{v_1} \subset \mathbb{C}^{v_2} \subset \cdots \subset \mathbb{C}^{v_n} \subset \mathbb{C}^w)$

$\circlearrowleft \textcircled{v} - \boxed{1} \rightsquigarrow$ Hilbert scheme of v points in $\mathbb{C}^2$

More natural to study $\bigsqcup_{\vec{v}} \mathcal{M}(\vec{v},\vec{w}) = \mathcal{M}(\vec{w})$

Fact: $\mathcal{M}_\theta(\vec{w}) \rightarrow \mathcal{M}_0(\vec{w})$ is a $\underline{\text{symplectic resolution}}$ (of singularities)

vast generalizations of Springer resolution

$G/B \rightarrow \mathcal{N}_B$

Geometric $\underline{\text{correspondences}}$ $\rightsquigarrow$ algebra action

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subvariety $Z \subset \mathcal{M} \times_{\mathcal{M}_0} \mathcal{M}$

push-pull using Z (in cohom, K-theory, Chow, $D^b Coh$, etc.)

$\bigsqcup_k Gr(k,N)$

e.g. $Gr(N) \times_{\mathcal{N}_P} Gr(N) \supset \{V_1 \subset V_2 \subset \mathbb{C}^N\} = Z$

$$\begin{array}{ccc} & Gr(N) \times_{\mathcal{N}_P} Gr(N) & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ Gr(N) & & Gr(N) \end{array}$$

$$p_{1*}\left(Z \otimes p_2^*(-)\right)$$

the associated operator

Everything can be done equivariantly wrt:

- $GL(\vec{w}) = \prod GL(w_i)$ $\circlearrowright$ framing nodes
- $\mathbb{C}^\times_\hbar$ $\circlearrowright$ scales the symplectic form

<u>Thm</u> [Nakajima, Okounkov, etc.]

$$\underbrace{K_{\mathbb{C}^\times_\hbar}\left(\mathcal{M}(\vec{w}) \times_{\mathcal{M}_0(\vec{w})} \mathcal{M}(\vec{w})\right)}_{\cong\; U_q(\hat{\mathfrak{g}}_Q)} \circlearrowright K_{\mathbb{C}^\times_\hbar}\left(\mathcal{M}(\vec{w})\right) = M(\vec{P})$$

$\mathfrak{g}_Q$ is some Lie alg. associated to Q. (e.g. if Q is a Dynkin diagram, $\mathfrak{g}_Q$ is the Lie alg)

Nakajima's standard modules associated to Drinfeld polys

$$P_i(z) = \prod_a (1-az)^{w_{i,a}}$$

eigenvalues of $\mathbb{C}^\times_\hbar$-action

dim of a-eigensp of W_i

with weight space decomposition:

$$\underset{M(\vec{P})}{K\left(\mathcal{M}(\vec{w})\right)} = \bigoplus_{\vec{v}} \underbrace{K\left(\mathcal{M}(\vec{v},\vec{w})\right)}_{= M(\vec{P})_{\vec{Q}}}$$

$$Q_i(z) = \prod_a (1-az)^{v_{i,a}}$$

(eigenvalues of $\psi_i^\pm$ are always of the form $\frac{P_i(zq)}{P_i(z/q)}\frac{Q_i(z/q)}{Q_i(zq)}$)

Many, many results follow:

① $M(\vec{P}) \twoheadrightarrow L(\vec{P})$ simple module

$\Rightarrow \{M(\vec{P})\}$ is a basis for $K(\mathrm{Rep}_{fin}\, U_q\hat{\mathfrak{g}})$

e.g. for $U_q\hat{sl}_2$, $M(P) = \bigotimes_{a_i \text{ roots of } P} V_1(a_i)$ generically, $= L(P)$

(Much simpler than $\{L(\vec{P})\}$, e.g.

$$M(\vec{P}^{(1)}) \otimes M(\vec{P}^{(2)}) = M(\vec{P}^{(1)}\vec{P}^{(2)}) \Big)$$

(2) $$\chi_q(M(\vec{P})) := \sum_{\vec{Q}} \dim\big(\underbrace{M(\vec{P})_{\vec{Q}}}_{K(\mathcal{M}(\vec{v},\vec{w}))}\big) \underbrace{Y_{\vec{P}} Y_{\vec{Q}}^{-1}}_{\text{formal monomial recording roots of } \vec{P} \,\&\, \vec{Q}}$$

$$= \sum_{Q} \chi(\mathcal{M}(\vec{v},\vec{w})) Y_{\vec{P}} Y_{\vec{Q}}^{-1}$$

e.g. if $GL(w)$ action on $\mathcal{M}$ has isolated fixed points, T

$\chi = \#(\text{fixed point})$.

great insight by Nakajima

$\Rightarrow$ geometric realization of χ_q on all of $\mathrm{Rep}_{fin}\, U_q\hat{\mathfrak{g}}$.

(3) $\exists$ refinement $$\chi_{q,t}(M(\vec{P})) = \sum_{\vec{Q}} \sum_k (-t)^k \dim H^k(\mathcal{M}(\vec{v},\vec{w})) Y_{\vec{P}} Y_{\vec{Q}}^{-1}$$

(q,t)-character

Is an injective hom. $K(\mathrm{Rep}_{fin}\, U_q\hat{\mathfrak{g}})[t^{\pm}] \hookrightarrow \mathbb{Z}[Y_{\vec{R}}^{\pm}, t^{\pm}]$

up to some shifts in t

(4) $\exists$ involution $\bar{t} = t^{-1}$ (up to the same shifts in t)

$\overline{Y_{\vec{R}}} = Y_{\vec{R}}^{-1}$

restricting to $K(\mathrm{Rep}_{fin})$ and:

Thm (Nakajima): "Kazhdan-Lusztig basis" of $K(\mathrm{Rep}_{fin}\, U_q\hat{\mathfrak{g}})[t^{\pm}]$ restricts at $t=1$ to $\{L(\vec{P})\} \subset K(\mathrm{Rep}_{fin})$

$\{\tilde{L}(\vec{P})\}$ s.t. $\overline{\tilde{L}(\vec{P})} = \tilde{L}(\vec{P})$

$$\tilde{L}(\vec{P}) \in M(\vec{P}) + \sum_{\vec{Q}<\vec{P}} t^{-1}\mathbb{Z}[t^{-1}] M(\vec{Q})$$

What is the change of basis $\{L\} \to \{M\}$?

Explicitly computable: take $\chi_{q,t}(M(\vec{P}))$ & add terms $t^{-1}\mathbb{Z}[t^{-1}]\,\underline{\chi_{q,t}(M(\vec{Q})}$) until you get something fixed under involution.