BGG Resolutions for finite-dimensional modules

$\boxed{16.1}$

Def: for $L(\lambda)$, a BGG resolution has the form

$$0 \to C_m \to C_{m-1} \to \dots \to C_1 \to C_0 = M(\lambda) \xrightarrow{\varepsilon} L(\lambda) \to 0$$

$C_m = M(w_0 \cdot \lambda)$ $\quad w_0 \in W$

$m = \ell(w_0) = |\Phi^+|$ $\quad$ ↑ Strong BGG

~~$C_0 = M(\lambda) \to L(\lambda)$~~

$$C_k = \bigoplus_{w \in W^{(k)}} M(w \cdot \lambda)$$

$W^{(k)} = \{w \in W \mid \ell(w) = k\}$

Note: categorification of Weyl Char. formula

$$\operatorname{ch} L(\lambda) = \sum_{w \in W^{(k)}} (-1)^{\ell(w)} \operatorname{ch}(M(w\cdot\lambda))$$

$\boxed{16.2}$ Construct something (Weak BGG)

Thm: $\lambda \in \Lambda^+$. There is an exact sequence

$$0 \to M(w_0 \cdot \lambda) = D_m^\lambda \to D_{m-1}^\lambda \to \dots \to D_1^\lambda \to D_0^\lambda \to L(\lambda) \to 0$$

$D_0^\lambda = M(\lambda)$

where each D_k^λ has a std. filtration w/ $M(w\cdot\lambda)$ appearing once for w w/ $\ell(w) = k$

Sketch:

- start with $\lambda = 0$ (will translate/shift later)
- consider $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{n}^-$ as a $\mathfrak{h}$-mod m-dim'l VS & associated modules $\Lambda^k(\mathfrak{g}/\mathfrak{b})$ $0 \le k \le m$

- basis of $\mathfrak{g}/\mathfrak{b}$ are cosets of $y_1, \ldots, y_m \in \mathfrak{n}^-$
- weights are negative roots
  & weights of $\mathfrak{h}$ on $\Lambda^k(\mathfrak{g}/\mathfrak{b})$ are the sums of distinct roots
- form modules

$$D_k = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \Lambda^k(\mathfrak{g}/\mathfrak{b})$$

- Each D_k has a std. filtration by the fact

(3.6): M f-d $\mathcal{U}(\mathfrak{g})$-mod, λ weight,

then $T = M(\lambda) \otimes M$

has a fin. filtration with quotients

isom. to $M(\lambda + \mu)$ for $\mu \in \mathfrak{h}^*$

occur $\dim M_\mu$ times

- $\Lambda^0(\mathfrak{g}/\mathfrak{b})$ trivial $\mathfrak{b}$-mod $\mathbb{C}$

$\Rightarrow D_0 = M(0)$

- $\Lambda^m(\mathfrak{g}/\mathfrak{b})$ 1-dim'l w/ weight

$$-\sum_{\alpha > 0} \alpha = 2\rho = w_0 \cdot 0$$

- Next, think about homs

$$\partial_k : D_k \to D_{k-1}$$

$$\varepsilon : M(0) \to L(0)$$

(c.f. Weibel 7.7)

$$C_k = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \Lambda^k \mathfrak{g}$$

$\uparrow$ free, left $\mathcal{U}(\mathfrak{g})$-mods

$\Lambda^0 \mathfrak{g} = k \qquad C_1 = \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$

define $\varepsilon : C_0 \to k$ augmentation map

$i(\mathfrak{g}) \mapsto 0$

corresponding ideal $I = \ker \varepsilon$

2-sided

$d : C_1 \to C_0$

$d(u \otimes x) = ux$

$$\Rightarrow \quad C_1(\mathfrak{g}) \xrightarrow{d} C_0(\mathfrak{g}) \xrightarrow{\varepsilon} k \to 0$$

$\uparrow$ exact

$d \geq 2 \quad d : C_i \to C_{i-1}$

Cartan

$d(u \otimes x_1 \wedge \dots \wedge x_k) =$

$$\sum_{i=1}^{k} (-1)^{i+1} u x_i \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_k$$
$$+ \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \dots \wedge x_k$$

$p = 2 \quad d(u \otimes x \wedge y) = ux \otimes y - uy \otimes x - u \otimes [x, y]$

- Relative version for us: $\Lambda^k \mathfrak{g} \rightsquigarrow \Lambda^k(\mathfrak{g}/\mathfrak{h})$

Def. maps: $u \in U(\mathfrak{g})$, representatives $z_1, \dots, z_k \in \mathfrak{g}$ of cosets $\xi_i \in \mathfrak{g}/\mathfrak{h}$

$\partial_k : D_k \to D_{k-1}$

$$\partial_k(u \otimes \xi_1 \wedge \dots \wedge \xi_k) = \sum_{i=1}^{k} (-1)^{i+1} (u z_i \otimes \xi_1 \wedge \dots \wedge \hat{\xi}_i \wedge \dots \wedge \xi_k)$$
$$+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} (u \otimes \overline{[z_i, z_j]} \wedge \dots \hat{\xi}_i \wedge \dots \hat{\xi}_{j} \dots \wedge \xi_k)$$

coset

Check: indep. of choice of representatives

also can check it's exact

by Artinian arguments

- D_k is "too large": cut off parts not in $\mathcal{O}^\lambda$
- $L(0) \rightsquigarrow L(\lambda)$

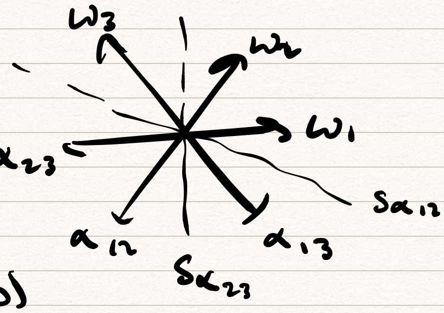
 idea: tensor resolution w/ $L(\lambda)$

 - $\otimes\, L(\lambda)$ is exact $\Rightarrow$ resolution of $L(\lambda)$

 $D_k^\lambda = (D_k^0 \otimes L(\lambda))^{\chi_\lambda}$

$\mathfrak{sl}_3(\mathbb{C})$

$W = S_3 = \langle s_{\alpha_{12}}, s_{\alpha_{23}} \rangle$



$s_{\alpha_1\alpha_2\alpha_3}(\lambda+\rho)-\rho$

$0 \to M(s_{\alpha_1\alpha_2\alpha_3}\cdot\lambda) \to M(s_{\alpha_{12}}s_{\alpha_{23}}\cdot\lambda) \oplus M(s_{\alpha_{23}}s_{\alpha_{12}}\cdot\lambda)$
$\to M(s_{\alpha_{12}}\cdot\lambda) \oplus M(s_{\alpha_{23}}\cdot\lambda) \to M(\lambda) \to L(\lambda) \to 0$

$\mathfrak{g}/\mathfrak{b}$ $\qquad f_{12} \quad f_{13} \quad f_{23}$

$$f_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{6.5}$ Some stuff about Ext

Thm: $\lambda \in \mathfrak{h}^*$

(1) if $\mathrm{Ext}_{\mathcal{O}}(M(\eta), M(\lambda)) \neq 0$ $\quad \eta \in \mathfrak{h}^*$

 then $\eta \uparrow \lambda$ but $\eta \neq \lambda$

(2) $\lambda \in \Lambda^+$, $w, w' \in W$

 if $\mathrm{Ext}_{\mathcal{O}}(M(w'\cdot\lambda), M(w\cdot\lambda)) \neq 0$

$\Rightarrow w < w'$ in Bruhat ordering

in particular $\ell(w) < \ell(w')$

Pf (1) Recall (Ch 3) if $\lambda, \eta \in \mathfrak{h}^*$, M highest weight module w/ weight η, $\lambda \neq \eta$, then

$$\mathrm{Ext}_{\mathcal{O}}(M(\lambda), L(\lambda)) = 0$$
$$\mathrm{Ext}_{\mathcal{O}}(M(\lambda), M(\lambda)) = 0$$

$\Rightarrow \lambda \neq \eta$

- Nonsplit SES

$$(*)\ 0 \to M(\lambda) \to M \to M(\eta) \to 0$$

$P(\eta) \to M(\eta)$ lifts to $P(\eta) \xrightarrow{\varphi} M$

- $\operatorname{im} \varphi \cap M(\lambda)$ nontrivial ($(*)$ would split)
- $P(\eta)$ has std. filtration

$$0 = P_0 \subset P_1 \subset \dots \subset P_n = P(\eta)$$
$$P_i / P_{i-1} \cong M(\eta_i) \quad \text{some } \eta_i$$

- BGG reciprocity & $[M(\lambda) : L(\eta)] \neq 0 \Rightarrow \eta \uparrow \lambda$

$\Rightarrow \eta \uparrow \eta_i$

- $\operatorname{im} \varphi \cap M(\lambda) \neq 0$, there is at least one i s.t. $\varphi(P_i) \cap M(\lambda) \neq 0$

$\Rightarrow M(\lambda)$ has a nonzero submod, which is a homomorphic image of $M(\eta_i)$

$\Rightarrow [M(\lambda) : L(\eta_i)] \neq 0$

$\xrightarrow{BGG} \eta_i \uparrow \lambda$, so $\eta \uparrow \eta_i \Rightarrow \eta \uparrow \lambda$

(2) (1) $\Rightarrow w' \cdot \lambda \uparrow w \cdot \lambda$

All of the linked weights for ws are regular integral

Strong linkage principal (5.2) : λ regular antidom.

$$[M(w\cdot\lambda) : L(w'\cdot\lambda)] \neq 0 \Rightarrow w' \leq w$$

flip inequality $\Rightarrow$ $w' > w$ for regular integral

Thm / Result / Fact / ... :

$\lambda \in \Lambda^+$, then weak BGG resolution is in fact a (strong) BGG resolution

idea: D_k^λ's split into a direct sum of vermas when thm above holds

§ BGG's Thm

Thm: $\lambda \in \Lambda^+$, then $\dim H^k(\mathfrak{n}^-, L(\lambda)) = |W^{(k)}|$

Pf: By definition, $H^k(\mathfrak{n}^-, L(\lambda)) = \mathrm{Ext}^k_{\mathfrak{n}^-}(\mathbb{C}, L(\lambda))$

Dualizing: $\mathrm{Ext}^k_{\mathfrak{n}^-}(\mathbb{C}, L(\lambda)) \underset{\text{as } \mathfrak{g}\text{-mods}}{\cong} \mathrm{Ext}^k_{\mathfrak{n}^-}(L(\lambda)^*, \mathbb{C})$

$$L(\lambda)^* \cong L(\underbrace{-w_0\cdot\lambda}_{\lambda^*})$$

Compute RHS: BGG, get resolution of $L(\lambda^*)$ by free $U(\mathfrak{n}^-)$-mods which are direct sums of vermas of the form $M(w\cdot\lambda^*)$

$\mathrm{Ext}^k_{\mathfrak{n}^-}(L(\lambda^*), \mathbb{C})$ is the cohom. of the complex $\mathrm{Hom}_{\mathfrak{n}^-}(M^\bullet, \mathbb{C})$

$$\left(M^\bullet = \text{complex with terms } \bigoplus_{w\in W^k} M(w\cdot\lambda^*) \right)$$

For any $\mathfrak{n}^-$-module M, we can identify

$$\mathrm{Hom}_{\mathfrak{n}^-}(M, \mathbb{C}) \hookrightarrow (M/\mathfrak{n}^- M)^*$$

lin maps which send $\mathfrak{n}^- M \to 0$

When $M = M(\mu)$, $M/\mathfrak{n}^- M \cong \mathbb{C}_\mu$ as $\mathfrak{h}$-mods

& dual $\mathbb{C}_{-\mu}$

$\Rightarrow$ k^{th} term of the complex $\mathrm{Hom}_{\mathfrak{n}^-}(M^\bullet, \mathbb{C})$

$$\cong \bigoplus_{w \in W^{(k)}} \mathbb{C}_{-w \cdot \lambda^*}$$

all distinct

$\Rightarrow$ all maps in our complex are zero

So $\mathrm{Ext}^k_{\mathfrak{n}^-}(L(\lambda^*), \mathbb{C})$ are just $\mathrm{Hom}_{\mathfrak{n}^-}(M^\bullet, \mathbb{C})$

Example: $\mathfrak{sl}_2(\mathbb{C})$

$\mathfrak{n}^- = \mathbb{C}$

$\mathcal{U}(\mathfrak{n}^-) = \mathbb{C}[x]$

t = trivial rep of $\mathbb{C}[x]$

Prop: (1) $\dim \mathrm{Hom}_{\mathbb{C}[x]}(t, L(n)) = 1$

(2) $\dim \mathrm{Ext}^1_{\mathbb{C}[x]}(t, L(n)) = 1$

(3) $\dim \mathrm{Ext}^i_{\mathbb{C}[x]}(t, L(n)) = 0 \quad i > 1$

(1) the rep of $\mathbb{C}[x]$ on $L(n)$ is given by sending $x \mapsto f \cdot L(n)$

$\Rightarrow$ $\dim \mathrm{Hom}_{\mathbb{C}[x]}(t, L(n))$ is the dim of the VS killed by f, i.e. lowest weight space

$\Rightarrow$ 1 dim'l

(3) $\mathbb{C}[x]$ is PID $\Rightarrow$ $\mathbb{C}[x]$ hereditary ring

so $\mathrm{Ext}^i(M, N) = 0$ $i > 1$ any two mods M, N

(2) ? Lie alg extensions