Cat. lecture

trans. funct. → Bernstein–Gelfand 1980
$\otimes$ of finite-dim & infin-dim rep. of simple L.A.
perverse sheaves on flag var

$\otimes$ modules over Hopf. alg, $U(\mathfrak{g})$ $\quad \mathcal{U} = \mathcal{U}(\mathfrak{g})$

$V \otimes M$ $\qquad$ $\mathcal{U}$-mod $\circlearrowleft$ $V\otimes$

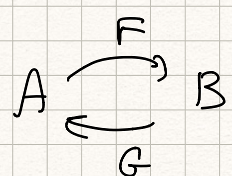
2 sided adjoint V^*

$\mathrm{Hom}_{\mathfrak{g}}(V\otimes M, N) \cong \mathrm{Hom}_{\mathfrak{g}}(M, V^*\otimes N)$.

biadjoint $V^{**} \cong V$

$\Updownarrow$

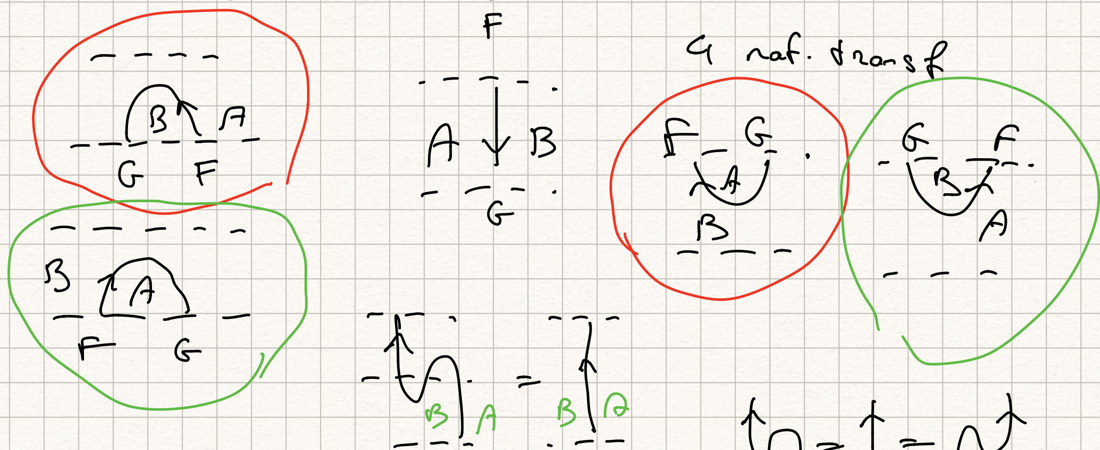
topology $\qquad (F, G)$ $\quad$ G both left/right adj to F

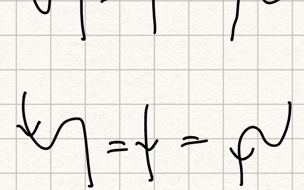


Poincare dual lang.

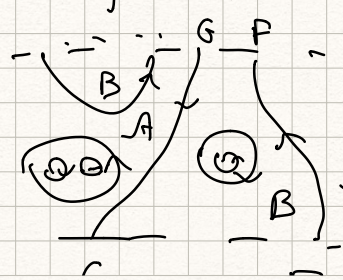
Id_F

4 nat. transf





Biadj. functors:

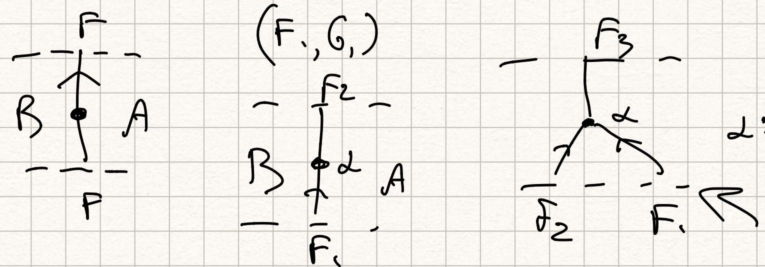


A $\Rightarrow$ nat. trans

TFT $\Rightarrow$

G F

planar diagrams $\xrightarrow{\text{a TFT}}$ nat. transformations



$\alpha : F_2 F_1 \Rightarrow F_3$

1) Cat. $\mathfrak{g}$ groups (Lauda $sl(2)$, $\angle$-k, R, CR)

2) Soergel cat, s. Soergel bimodules

$sl(n)$ B. Elias, G Williamson,

3) ...

2D TQFT (planar $\mathbb{R}^2$) with defects.

biadjoint functors.

hom-cat $\quad V\otimes ? \xleftrightarrow{\text{biadj}} V^* \otimes ?$ (Bernstein Trace formula)

$V\otimes \quad \mathcal{U}(\mathfrak{g})\text{-mod} \to \mathcal{U}(\mathfrak{g})\text{-mod}$ via planar diagrams

$\mathcal{O} \quad \mathfrak{g} = sl(n), gl(n)$

$\mathcal{O} = \bigoplus_{\nu \in Z^n} \mathcal{O}_\nu \qquad \frac{\mathcal{O}_\nu \simeq \text{rep. over f.d. alg.}}{\text{BGG} \quad \text{recip.}}$

simple, indec proj.

A-mod $\quad$ A-f.d. alg/k.

fin. many simples, bijection simples $\leftrightarrow$ indec. projectives

$\updownarrow$

$\mathcal{O}_\nu \simeq A_\nu$ ← writing A_ν is complicated | indec. injective

A - f.d $\rightsquigarrow$ B - f.d. morita equiv to A
"smallest"

$\underline{A\text{-mod} \simeq B\text{-mod}}$

(interesting Mor. equiv in ∞-dim alg Diff over singular alg. curves
Berest-Wilson)

$\mathcal{O}_\nu \to$ smallest B_ν

$\underline{B_\nu\text{-mod} \simeq \mathcal{O}_\nu.}$

$\underline{\mathcal{O}_{reg} \simeq A_{reg}\text{-mod}}$ $\qquad A_{reg} \leftarrow$ Soergel category

$\mathfrak{sl}(n)$ $\qquad \forall$ 2 reg. blocks are equivalent.

$\mathfrak{sl}(2)$ $\quad \circ \quad \circ$ $\qquad \updownarrow$

Γ $\qquad$ irreps of $\mathfrak{g}$

$$\begin{tikzcd} \underset{0}{\circ} \arrow[r, bend left] & \underset{1}{\circ} \arrow[l, bend left] \end{tikzcd}$$

path alg (Γ)

$\Bbbk\Gamma/(010)$ $\qquad (010)=0$ $\qquad A_2$

$$\begin{tikzcd} \underset{a}{\bullet} \arrow[r, bend right] & \underset{b}{\bullet} \arrow[l, bend right] \end{tikzcd}$$

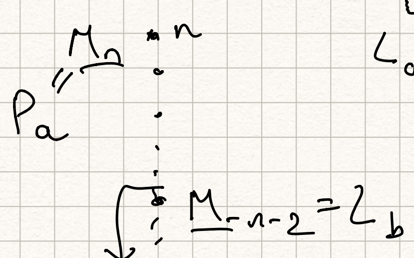
$(0), (1), (01), (10), (101).$

$1 = (0) + (1)$

$\underline{(aba) = 0.}$

$(a), (b), (ab), (ba)$
$(bab).$

$A_2\text{-mod} \simeq \underline{\mathcal{O}_{reg}(\mathfrak{sl}_2)}$



$L_a = P_a / L_b \leftarrow$ fin. dim dim $=1$

$M_n^\vee$ - dual verma

$\underline{P_{-n-2} = P_b}$

$\mathrm{End}(P_b) \simeq \underline{\Bbbk[x]/(x^2)}$ $\qquad x = (bab).$

$B = \Bbbk[x]/(x^2)$ $\quad M = B \oplus \underline{\Bbbk}$ trivial, x acts by 0.

$\mathcal{E}nd(M) \simeq A_2$. $\quad \underline{\Bbbk 1} \to \overset{B}{x} \to \overset{\Bbbk}{0}$

$\qquad\qquad\qquad\qquad 1 \to 1$

Principle $\forall$ f.d. alg is quotient of some path alg/rels.

A_n, $A_n^!$

$$\begin{tikzcd} \underset{0}{\circ} \arrow[r, bend right] & \underset{1}{\circ} \arrow[l, bend right] \arrow[r, bend right] & \underset{2}{\circ} \arrow[l, bend right] \arrow[r, bend right] & \circ \arrow[l, bend right] \arrow[r, bend right] & \underset{n-1}{\circ} \arrow[l, bend right] \end{tikzcd}$$

$(i|i+1|i+2) = 0 = (i|i-1|i-2)$ $\qquad (i|i+1|i) = (i|i-1|i)$

$(0|1|0) = 0$.

$sl(n)$

$R = \Bbbk[x_1 .. x_n]$ Soergel bimodules

$B_i = R \otimes_{R^i} R$ $\qquad R \otimes_{R^i} R/m$

$R^i = \Bbbk[x_1, .., x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, ..., x_n]$

Soergel bimodules

$R_i \subset R$

R free R_i-mod basis $\{1, x_i\}$.

$B_{\underline{w}} = B_{i_1} \otimes .. \otimes B_{i_m}$ $\quad w \in S_n$

Soergel bimodules $\underline{w} = s_{i_1} .. s_{i_m}$

$B = \bigoplus_{w \in S_n} B_{\underline{w}}$ $\quad \boxed{\begin{array}{c}\mathcal{E}nd(B, B) \\ R\otimes R \text{ ss-dim.}\end{array}}$ f.d/R $\quad R \otimes_{R^i} R = 1 \otimes_{R^i} R \oplus x_i \otimes_{R^i} R$

kill x's on one side

$R \hookrightarrow B_i \hookleftarrow R$

$R \supset m = (x_1 .. x_n)$

$B_i/m \neq R \otimes_{R^i} R/m = \Bbbk \cdot 1 \oplus \Bbbk \cdot x_i$

$B_i / B_i m$ $\quad \boxed{\tilde{B}_{\underline{\omega}} = B_{\underline{\omega}} / B_{\underline{\omega}} m}$ reduced Soergel bimodules

fin. dim

$\tilde{B} = \oplus \tilde{B}_{\underline{\omega}}$

$R.$

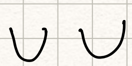
$\mathrm{End}_R(\tilde{B})\text{-mod} \simeq \mathcal{O}_{reg.}$

$\mathfrak{sl}(2)$ $\quad 1, s$

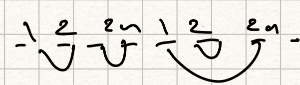
$R, \quad B_s = R \otimes_{R^s} R = \Bbbk[x] \otimes_{\Bbbk[x^2]} \Bbbk[x]$

$R = \Bbbk[x].$ $\quad x$

basic alg for a block is hard

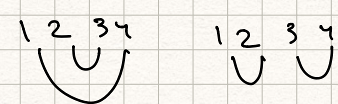


H^n



$a \quad b \quad P_a$

$1 = \sum_{a \in B^n} 1_a$



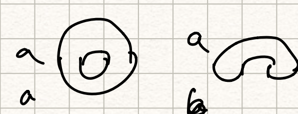
$n = 2$

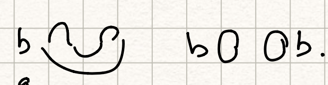
2D TQFT

Comm. Frob. alg

$A = \Bbbk[x]/(x^2).$

$\varepsilon(x) = 1, \quad \varepsilon(1) = 0.$





$\Downarrow F$

$A^{\otimes 2} \oplus A \otimes$

$\oplus A \oplus A^{\otimes 2}$

H^n - unital assoc. alg, f.d. $/\Bbbk$

H^n-mod $\dashrightarrow$ a piece of $\underline{\mathcal{O}}$ for $sl(2n)$.

reg. block $\supset$ parabolic subcat

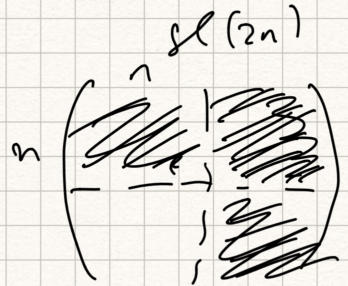
modules that are

sum. of f.d. mod

over

$P_{n,n}$ parabolic L. alg.

inf. cat



$M \in \mathcal{O}$ for $sl(2n)$, $\underline{M|_{P_{n,n}}}$

$\underline{\text{locally f.d. over } P_{n,n}}$

$$M \simeq \bigoplus_{i \in I} M^i$$

I a set

$$P_{n,n} \supset \underline{gl(n)^1} \oplus \underline{gl(n)^2}$$

f.d. mod

$sl(n)^1$ $gl(n)^2$

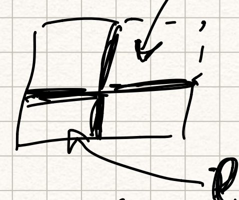
$$V_\lambda \simeq V_1 \otimes V_2$$

f.d.

acts by 0

extend by 0 to $P_{n,n}$

get a module over $\mathcal{U}(P_{n,n})$



$$\underline{M_V = \mathrm{Ind}_{\underline{\mathcal{U}(P_{n,n})}}^{\mathcal{U}(\mathfrak{g})} V} \simeq$$

size of M_V?

$$gl(2n) = P_{n,n} + \ell$$

$$\simeq S(\ell) \otimes V$$

$\underline{v} \in V_1 \otimes V_2$ $\quad$ $\mathfrak{p}_{n,n} v \subset V_1 \otimes V_2$.

$\forall \omega \in M_V$ $\quad$ $\dim \mathcal{U}(\mathfrak{p}_{n,n}) \underline{\underline{\omega}} < \infty$.

$t \in \mathfrak{p}_{n,n}$

$t(x_k \cdot x_1 \ldots)$ $\quad$ $x_i \in \mathfrak{l}$

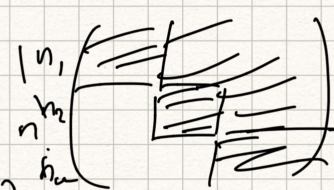
$\mathfrak{p} \subset \mathfrak{g}$

parabolic cat. $\mathcal{O}$.

$\mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$

$\uparrow$ define $\mathcal{O}$.

<u>Thm</u> <u>BGS</u> Koszul duality between <u>singular</u> <u>blocks</u> of $\mathcal{O}$ & <u>parabolic block</u> (parabolic subcat of reg. block).

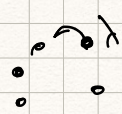


<u>singular blocks of $\mathcal{O}$</u>

integral weights. for $sl(n)$

stabilizer of weight under (shifted) action of W.

reg: weights $\Leftrightarrow$ el's of W.



<u>singular blocks</u> : nontrivial stabilizer

$\bullet \lambda$ $\quad$ $Stab(\lambda) \simeq S_{n_1} \times S_{n_2} \times \ldots \times S_{n_k} \subset S_n$

$$|W \cdot \lambda| = \binom{n}{n_1, n_2 \ldots n_k}.$$

for this sing. block $\mathcal{O}_\lambda$. $\quad$ $\mu \in W \cdot \lambda$.

$K_0(\mathcal{O}_\lambda)$ - free. ab., basis $[M_\mu]$

$\mathcal{O}$ sing. block

$\mathcal{O}_\lambda$ trans. functions Many trans functions.

comp. of t. functions

in $\mathcal{O}$ $V\otimes$ has many dir. summands

$pr_{\chi_2}(V\otimes pr_{\chi_1} M)$

$\mathcal{O}_\lambda$ λ - integral

$\mathcal{O}_{\underline{\lambda}} \overset{T}{\longleftrightarrow} \mathcal{O}_{\underline{\mu}}$ $\underline{\operatorname{Stab}\lambda = \operatorname{Stab}\mu}$ biadjoint

$\mathcal{O}_{reg}$ $T_i : \mathcal{O}_{reg} \xrightarrow{F_i} \mathcal{O}_i \xrightarrow{F_i'} \mathcal{O}_{reg}.$

i-wall

$[T_i] = \underline{1+s_i}$

$[T_i M] = (1+s_i)[M]$ S_n

$K_0(\mathcal{O}_{reg}) \simeq \mathbb{Z}[S_n]$

$\cup$ group ring of S_n

"pos. one"

$\underline{\underline{s_i}}$

$\mathbb{Z}[S_n]$ acts on itself. $\underline{\mathbb{Z}[S_n]}$

ring module of rank 1

after cat, we want to see an action of

a monoidal cat (Soergel cat) on a module.

Soergel cat $\circlearrowright$ $\mathcal{O}_{reg.}$ ← not a monoidal

$\Downarrow$ Groth. gr.

$\mathcal{H}_n \circlearrowright \mathcal{H}_n$

$\mathcal{H}_n \circlearrowright$ [Induced rep]

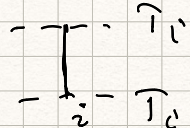
Hecke alg $\mathbb{Z}[q,q^{-1}]$

$\mathbb{Z}(S_n) \circlearrowright \mathbb{Z}[S_n]$

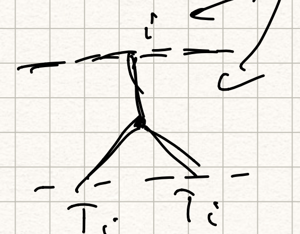
B_ω $\quad$ B_ω / m $\quad$ [Infin char / gen. char]

T_i – translation functors. $\quad 1+s_i$

$\mathrm{Hom}(T_{i_k} \ldots T_{i_1}, T_{j_m} \ldots T_{j_1})$ $\quad$ BD Graphs



$T_i \otimes T_i \simeq T_i \otimes T_i$



B. Elias

$sl_k \circlearrowright \mathcal{O}$(sing blocks) $\longrightarrow$ Cat. of [$U(g)$, $U_q(g)$.]

by trans functions

$\mathcal{H}_n$ $\quad$ indec. Soergel bimodules – KL. basis

indec. functors $\quad B^{ind}_\omega$

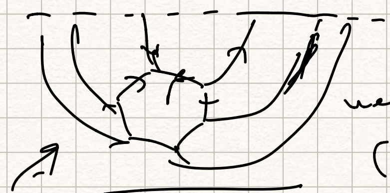
$A\text{-mod} \simeq \mathcal{O}_{reg}$

$A'\text{-mod}$ basis $\quad B^{ind} = \bigoplus_\omega B^{ind}_\omega \quad \mathrm{End}(B^{ind}) \simeq A'$

+ max. ideal

$sl(3)$

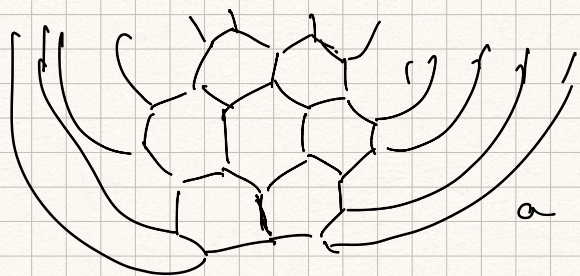
H^n



idemp come from reduced webs w. boundary

(non-ell. webs Kuperberg)

For a while, get indec. proj. modules



a

2D 3D

P_a – decomposable