TRANSLATION FUNCTORS IN CATEGORY $\mathcal{O}$

NIKOLAY GRANTCHAROV

Abstract. These notes are for a talk in a student seminar on Category $\mathcal{O}$. Translation functors are defined by tensoring with a finite dimensional $\mathfrak{g}$-representation and then projecting to a block. Our goal is to describe how these functors act on integral blocks. We show they provide an equivalence of categories between all regular integral blocks, and then describe the result of translating from regular integral blocks to blocks parameterized by weights lying on walls.

1. Notation and Setup

We use the following setup:

- Fix a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbf{C}$.
- Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.
- Fix simple roots $\alpha_i \in \Delta$ and positive roots Φ^+ inside root system $\Phi \subset \mathfrak{h}^*$ corresponding to above choice of $\mathfrak{b}$.
- Let E be the Euclidean space spanned by Φ.
- Let $\Lambda := \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbf{Z} \text{ for all } \alpha \in \Phi\}$ be the integral weight lattice.
- Let $\Lambda^+ := \{\lambda \in \Lambda : \langle \lambda, \alpha^\vee \rangle \in \mathbf{Z}_{\geq 0} \text{ for all } \alpha \in \Phi^+\}$ be the dominant integral weight lattice.
- Write W for Weyl group associated to $\mathfrak{g}$. It acts on $\mathfrak{h}^*$ the standard way by $s_{\alpha_i}(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$, where s_{α_i} is simple reflection associated to simple root α_i. These generate W. The Weyl group also acts in a ρ-twisted way by the *dot action*: $w \cdot \lambda := w(\lambda + \rho) - \rho$. Write the stabilizer of λ via the dot action as $W_\lambda := \mathrm{Stab}_W(\lambda) := \{w \in W : w \cdot \lambda = \lambda\}$. A weight is *regular* if $\mathrm{Stab}_W(\lambda) = \{\mathrm{Id}\}$.
- Denote by Verma modules $M(\lambda)$ and unique simple quotient by $L(\lambda)$.
- There is a decomposition of categories $\mathcal{O} = \oplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\lambda$ parameterized by central characters $\chi_\lambda : Z\mathfrak{g} \to \mathbf{C}$. When $\lambda \in \Lambda$, the subcategories $\mathcal{O}_\lambda$ are blocks, in particular indecomposable, and are built from the simple objects $L(w \cdot \lambda)$. In particular, there are $|W/\mathrm{Stab}_W(\lambda + \rho)|$ simple objects in $\mathcal{O}_\lambda$.

We emphasize that all weights considered are integral. The main results may be formulated for nonintegral weights, and the proofs are nearly identical. However we choose to omit nonintegral weights as this forces cumbersome notation. Moreover, integral weights are the ones relevant for study of algebraic groups since they exponentiate to characters of the torus, and so all our results may be lifted to the G semisimple simply-connected case.

Our exposition will somewhat follow [Hum08, Ch. 7], and the proofs given are essentially the original proofs of Jantzen in [Jan, Ch. 2]

2. Basic Properties of Translation Functors

Let $\lambda, \mu \in \Lambda$ be integral weights and let $\nu := \mu - \lambda$. Since the usual action of W acts freely and transitively, there is unique $w \in W : \bar{\nu} := w\nu \in \Lambda^+$.

Definition 2.0.1. *Let $\lambda, \mu \in \Lambda$ and $\bar{\nu}$ as above. Define a translation functor as*

$$T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu, \quad T_\lambda^\mu M := pr_\mu(L(\bar{\nu}) \otimes M).$$

Date: September 18, 2020.

Proposition 2.0.1. *The translation functor T_λ^μ is exact, commutes with duality, and takes projectives to projectives.*

Proof. Recall tensoring an object in category $\mathcal{O}$ with a finite-dimensional representation lands in category $\mathcal{O}$, and it defines an exact functor. Moreover, an exact sequence of $\mathfrak{g}$-modules preserves weight spaces, hence projecting to μ block is also exact. The statement about duality follows from the easy facts: $(L(\bar{\nu})\otimes M)^\vee = L(\bar{\nu})^\vee \otimes M^\vee$, $L(\bar{\nu})^\vee \cong L(\bar{\nu})$, and $(M^\vee)^\chi \cong (M^\chi)^\vee$. Finally, projective tensor finite-dimensional is projective, and projecting to summand it still remains projective. ☐

Proposition 2.0.2. *The functor T_λ^μ is left and right adjoint to T_μ^λ.*

Proof. By symmetry, it's enough to prove

$$\mathrm{Hom}_\mathcal{O}(T_\lambda^\mu M, N) \cong \mathrm{Hom}_\mathcal{O}(M, T_\mu^\lambda N).$$

Recall $L(\bar{\nu})^* \cong L(-w_0\bar{\nu})$, where $w_0 \in W$ is longest element. We see $-w_0\bar{\nu} = -w_0 w\nu = w_0 w(-\nu) \in W(-\nu) \cap \Lambda^+ \Rightarrow -w_0\bar{\nu}$ is dominant weight conjugate to $-\nu$. We finish by using usual tensor-hom adjunction (for Category $\mathcal{O}$): For L finite-dimensional,

$$\mathrm{Hom}_\mathcal{O}(L\otimes M, N) \cong \mathrm{Hom}_\mathcal{O}(M, L^* \otimes N). \tag{2.0.1}$$

☐

Finally, we recall the following using lemma:

Lemma 2.0.1. *Let L be finite-dimensional. Then $M(\lambda)\otimes L$ admits a standard filtration by Verma modules, with the multiplicity*

$$\mathit{mult}(M(\lambda+\mu), M(\lambda)\otimes L) = \dim L_\mu$$

Note, given λ, μ in the dominant Weyl chamber, it is easy to compute the character of the translation functor: If $V \in \mathcal{O}_\lambda$ is finite dimensional and $\mathrm{Ch}V = \sum_{w\in W} a_w \chi(w\cdot\lambda)$ for $a_w \in \mathbf{Z}$, then

$$\mathrm{ch}T_\lambda^\mu L(\lambda) = \sum_{w\in W} a_w \sum_{w_1 \in \mathrm{Stab}_W(\lambda)/(\mathrm{Stab}_W(\lambda)\cap \mathrm{Stab}_W(\mu))} \chi(ww_1\cdot\mu).$$

However, it is hard in general to describe the composition factors, even in the case V is simple. We explore this in the subsequent sections by considering the "degeneracy" (i.e how many walls it lies on) of a weight λ.

3. Geometry of Weyl Group

Recall that *Weyl Chambers* in E are the connected components of the complement of the union of hyperplanes orthogonal to the (positive) roots. In particular, these are open in Euclidean topology and Zariski dense. Since we are working with blocks parameterized by weights $\lambda \in \mathfrak{h}^*$ modulo the W-dotted action, we must shift our origin 0 to $-\rho = -\sum_{\alpha\in\Phi^+}\alpha$ and hyperplanes to

$$H_\alpha := \{\lambda \in E : \langle \lambda+\rho, \alpha^\vee\rangle = 0\}.$$

However, we must refine Weyl chambers to allow for walls. This is done via the following definition

Definition 3.0.1. *Define facet F as follows: Decompose $\Phi^+ = \Phi_F^+ \sqcup \Phi_F^0 \sqcup \Phi_F^-$ and require:*

$$\lambda \in F \Leftrightarrow \begin{cases} \langle \lambda+\rho, \alpha^\vee\rangle = 0 & \textit{when } \alpha \in \Phi_F^0 \\ \langle \lambda+\rho, \alpha^\vee\rangle > 0 & \textit{when } \alpha \in \Phi_F^+ \\ \langle \lambda+\rho, \alpha^\vee\rangle < 0 & \textit{when } \alpha \in \Phi_F^-. \end{cases}$$

We define the closure $\bar{F}$ by removing the strictness in the inequalities, and define the upper closure

$$\lambda \in \hat{F} \Leftrightarrow \begin{cases} \langle \lambda+\rho, \alpha^\vee\rangle = 0 & \text{when } \alpha \in \Phi_F^0 \\ \langle \lambda+\rho, \alpha^\vee\rangle > 0 & \text{when } \alpha \in \Phi_F^+ \\ \langle \lambda+\rho, \alpha^\vee\rangle \leq 0 & \text{when } \alpha \in \Phi_F^-. \end{cases}$$

Example For $\mathfrak{g} = \mathfrak{sl}_3$, there are 13 facets. 6 come from the open Weyl chambers, 6 come from the half lines spanned by root hyperplanes, and 1 comes from the origin, $-\rho$.

Observe, the Weyl chambers C are precisely the facets for which $\Phi^0_C = \emptyset$, and the closure $\bar{C}$ is a fundamental domain of W-dotted action. The facets F of $\bar{C}$ for which $|\Phi^0_F| = 1$ are called *walls*. Each walls lies on either upper or lower closure of C. Next observe the for a given facet F, the reflections corresponding to roots $\pm\alpha \in \Phi^0_F$ generate a subgroup of W consisting of elements which fix F pointwise. Next, observe each facet lies in the upper closure of a unique Weyl chamber. Finally, we observe the Euclidean space E is a disjoint union of the upper closures of chambers.

We conclude this section with an important lemma which will be used several times.

Lemma 3.0.1. *Suppose $\lambda \in F$ for some facet F and $\mu \in \bar{F}$. Let $\nu = \mu - \lambda$ and $\bar{\nu}$ be the unique W conjugate lying in Λ^+. Then for all other weights $\nu' \neq \nu$ of $L(\bar{\nu})$, the weight $\lambda + \nu'$ is not linked by W to $\lambda + \nu = \mu$.*

Proof. Suppose the given facet F is contained in the closure of some chamber $\bar{C}$. Given two chambers C, C', we define the distance $d(C, C')$ to be the number of root hyperplanes H_α separating C from C'. Suppose for contradiction there is some ν' and $w \in W : w \cdot (\lambda + \nu') = \lambda + \nu$, and consider the $\nu' \neq \nu$ weight of $L(\bar{\nu})$ which minimizes $d(C, C')$, where $\overline{C'}$ is the closure of chamber containing $\lambda + \nu'$. We induct on the distance.

If $d(C, C') = 0$, then by definition $C = C'$. But $\overline{C'}$ is fundamental domain for W-dotted action and $\lambda + \nu' \neq \lambda + \nu$, contradiction.

Suppose $d(C, C') > 0$. Then there exists hyperplane H_α which separates C' and C and $H_\alpha \cap C'$ contains a wall of C'. Say C' lies on positive side and C on negative: in particular

$$\xi \in \bar{F} \subset \bar{C} \Rightarrow \langle \xi + \rho, \alpha^\vee \rangle \leq 0.$$

Write $C'' := s_\alpha \cdot C'$. Since C is separated from C'' by same hyperplanes except H_α which separate C from C',

$$d(C, C'') < d(C, C').$$

Now,

$$\lambda + \nu \in \bar{C}' \Rightarrow \langle \lambda + \nu' + \rho, \alpha^\vee \rangle \geq 0 \Rightarrow s_\alpha \cdot (\lambda + \nu') = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha + s_\alpha \nu' \leq \lambda + \nu'$$

and combining this with $\lambda \in F \Rightarrow \langle \lambda + \rho, \alpha^\vee \rangle \leq 0$, we deduce

$$s_\alpha \nu' \leq s_\alpha \nu' - \langle \lambda + \rho, \alpha^\vee \rangle \alpha \leq \nu'$$

Set $\nu'' := s_\alpha \nu' - \langle \lambda + \rho, \alpha^\vee \rangle \alpha$. Then we find

$$s_\alpha \cdot (\lambda + \nu') = \lambda + \nu'' \in s_\alpha \cdot \overline{C'} = \overline{C''}. \tag{3.0.1}$$

And since $\nu, s_\alpha \nu$ are weights of $L(\bar{\nu})$ with $s_\alpha \nu' \leq \nu'' \leq \nu'$ and ν'' being in same α string, we conclude ν'' is also a weight of $L(\bar{\nu})$. But the minimality assumption on ν', and 3.0.1, and induction hypothesis together imply $\nu'' = \nu$. Then $s_\alpha \nu' \leq \nu'' \leq \nu'$ makes $s_\alpha \nu' = \nu$ since we cannot have both $\nu + \alpha$ and $\nu - \alpha$ weights of $L(\bar{\nu})$ (since $\bar{\nu} = w\nu$ and $w(\nu \pm \alpha) = \bar{\nu} \pm w\alpha < \bar{\nu}$ contradiction).

Thus, $\nu = \nu''$ and $s_\alpha \nu' = \nu$ forces $\langle \lambda + \rho, \alpha^\vee \rangle = 0$. And since $\lambda \in F$, all $\xi \in \overline{F}$ also satisfy $\langle \xi + \rho, \alpha^\vee \rangle = 0$. In particular this holds for $\xi = \lambda + \nu \in \overline{F} \Rightarrow \langle \nu, \alpha^\vee \rangle = 0$ and we conclude from $s_\alpha \nu' = \nu$ that $\nu = \nu'$, contradiction. □

4. Translation within a Facet

In the case a translation functor does not cross walls, i.e λ, μ lie in same facet, then there is an equivalence of categories:

Theorem 4.1. *Suppose $\lambda, \mu \in \Lambda$ belong to same facet F. Then $T^\mu_\lambda, T^\lambda_\mu$ provide an equivalence of categories between Grothendieck groups $K(\mathcal{O}_\lambda)$ and $K(\mathcal{O}_\mu)$, sending $[M(w \cdot \lambda)] \mapsto M(w \cdot \mu)]$ and $[L(w \cdot \lambda)] \mapsto [L(w \cdot \mu)]$.*

Proof. Recall $[M(w\cdot\lambda)]$ and $[L(w\cdot\lambda)]$ form a $\mathbf{Z}$ basis of $K(\mathcal{O}_\lambda)$. By theorem 5.1 (proved independently), $T_\lambda^\mu M(w\cdot\lambda)=M(w\cdot\mu)$ and inversely for T_μ^λ. Thus, for an arbitrary basis element M of $\mathcal{O}_\lambda$, written uniquely as $\mathbf{Z}$ linear combination of $[M(w\cdot\lambda)]$, then $T_\lambda^\mu\circ T_\mu^\lambda:[M]\mapsto[M]$ gives isomorphism of Grothendieck groups. Since $[L(w\cdot\lambda)]$ also form basis of $K(\mathcal{O}_\lambda)$ and the composition must map simple to either simple or 0, the statement follows. □

Now, we can formally deduce an equivalence of categories on Grothendieck group implies equivalence of categories by 6.0.1. Instead we prove it directly in our case.

Theorem 4.2. *Suppose $\lambda,\mu\in\Lambda$ belong to same faceet F. Then $T_\lambda^\mu, T_\mu^\lambda$ provide an equivalence of categories between $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$, and the functor $T_\mu^\lambda\circ T_\lambda^\mu$ is isomorphic to pr_λ.*

Proof. We must show $T_\mu^\lambda\circ T_\lambda^\mu$ is naturally isomorphic to the identity on $\mathcal{O}_\lambda$. We induct on the length of M. By adjointness, there is isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(T_\mu^\lambda\circ T_\lambda^\mu M, M)\cong \mathrm{Hom}_{\mathcal{O}}(T_\lambda^\mu M, T_\lambda^\mu M)$$

Let ϕ_M correspond to identity on right. Hence by induction, the first and third vertical arrows are isomorphisms,

$$\begin{tikzcd}
0 \arrow[r] & T_\mu^\lambda\circ T_\lambda^\mu N \arrow[r] \arrow[d, "\phi_N"] & T_\mu^\lambda\circ T_\lambda^\mu M \arrow[r] \arrow[d, "\phi_M"] & T_\mu^\lambda\circ T_\lambda^\mu L \arrow[r] \arrow[d, "\phi_L"] & 0 \\
0 \arrow[r] & N \arrow[r] & M \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

and by five lemma we conclude. □

5. Translation from facet to boundary

Theorem 5.1. *Let $\lambda,\mu\in\Lambda$ and assume λ lies in a facet F and $\mu\in\overline{F}$. Then*

$$T_\lambda^\mu M(w\cdot\lambda)\cong M(w\cdot\mu),\quad \textit{for all } w\in W$$

Proof. By lemma 2.0.1, $T_\lambda^\mu M(w\cdot\lambda)$ has filtration by $M(w\cdot\lambda+\nu')$ where ν' appears with multiplicity $\dim L(\bar\nu)_{\nu'}$. Thus, $M(w\cdot\mu)=M(w\cdot\lambda+\nu)$ occurs with multiplicity once. Moreover, lemma 3.0.1 applied to $w\cdot\lambda$ and $w\cdot\mu$ shows no other Verma's appear. □

Corollary 5.0.1. *Let $\lambda,\mu\in\Lambda$ and assume λ lies in a facet F and $\mu\in\overline{F}$. Then either $T_\lambda^\mu L(w\cdot\lambda)\cong L(w\cdot\mu)$ or 0.*

Proof. There are maps $M(w\cdot\lambda)\twoheadrightarrow L(w\cdot\lambda)\hookrightarrow M(w\cdot\lambda)^\vee$. Then by exactness commutativity with duality of T_λ^μ:

$$M(w\cdot\mu)\twoheadrightarrow T_\lambda^\mu L(w\cdot\lambda)\hookrightarrow M(w\cdot\mu)^\vee.$$

So the middle term is highest weight module with weight $w\cdot\mu$ or 0, and the injection implies if nonzero, then it is $L(w\cdot\mu)$. □

Example For $\mathfrak{g}=\mathfrak{sl}_2$, we can use theorem 5.1 to show $T_m^n L(m)=0$ if $m\in\mathbf{Z}_{\geq 0}$ and $n=-1=-\rho$. On the other hand, $T_m^n L(-m-2)=L(n)$, so both cases of above corollary are possible.

Theorem 5.2. *Let $\lambda,\mu\in\Lambda$ with $\lambda\in F$ and $\mu\in\overline{F}$. Then*

$$T_\lambda^\mu L(w\cdot\lambda)\cong\begin{cases} L(w\cdot\mu) & \textit{if } w\cdot\mu\in\widehat{w\cdot F}\\ 0 & \textit{else.}\end{cases}$$

Proof. By above corollary, we know the image is either $L(w\cdot\mu)$ or 0, and we know $T_\lambda^\mu M(w\cdot\lambda) = M(w\cdot\mu)$. Thus T_λ^μ must take some composition factor $L(w'w\cdot\lambda)$ of $M(w\cdot\lambda)$ to $L(w\cdot\mu)$, where $w'w\cdot\lambda \le w\cdot\lambda$. And exactness of T_λ^μ implies no other composition factors are sent to $L(w\cdot\mu)$. Thus $T_\lambda^\mu L(w'w\cdot\lambda)\cong L(w'w\cdot\mu)$ implies $w'w\cdot\mu = w\cdot\mu$, hence w' lies in subgroup W' generated by $s_\alpha : \alpha\in w\Phi_F^0$.

Suppose $w\cdot\mu\in\widehat{w\cdot F}$. Then for $\alpha\in w\Phi_F^0$, we claim $s_\alpha w\cdot\lambda\ge w\cdot\lambda$. Indeed, for such α, we find $\langle w(\mu+\rho),\alpha^\vee\rangle = 0\Rightarrow\langle w(\lambda+\rho),\alpha^\vee\rangle\le 0\Rightarrow s_\alpha w\cdot\lambda\ge w\cdot\lambda$. Thus since W' is Weyl group for root system Φ_F^0, we conclude $w'w\cdot\lambda\ge w\cdot\lambda$ for all $w'\in W'$. By above paragraph, we conclude equality and hence $L(w\cdot\lambda)$ is unique composition factor taken to $L(w\cdot\mu)$.

Suppose $w\cdot\mu\notin\widehat{w\cdot F}$. Then there is some hyperplane for s_α bounding $w\cdot F$ below. Namely, $s_\alpha w\cdot\lambda\le w\cdot\lambda$. By Verma's theorem, there is inclusion $M(s_\alpha w\cdot\lambda)$ into $M(w\cdot\lambda)$. But T_λ^μ maps both into $M(w\cdot\mu)$, since $s_\alpha w\cdot\mu = w\cdot\mu$. Thus it maps the quotient $Q := M(w\cdot\lambda)/M(s_\alpha w\cdot\lambda)$ to 0, which thus implies the quotient $T_\lambda^\mu L(w\cdot\lambda)$ of $T_\lambda^\mu Q$ is 0 □

6. Translation from boundary to interior

Recall given $\lambda\in\Lambda$, we denoted by W_λ for the stabilizer of λ in W with the dot action.

Theorem 6.1. *Let $\lambda,\mu\in\Lambda$ with $\lambda\in F$ and $\mu\in\overline{F}$. Then for all $w\in W$,*

$$chT_\mu^\lambda M(w\cdot\mu) = \sum_{w'\in W_\mu/W_\lambda} chM(ww'\cdot\lambda)$$

In particular, all Verma's occur as quotients in a standard filtration of $T_\mu^\lambda M(w\cdot\mu)$ with multiplicity 1.

Proof. We see

$$\begin{aligned}(T_\mu^\lambda M(w\cdot\mu) : M(ww'\cdot\lambda)) &= \dim\mathrm{Hom}_{\mathcal{O}}(T_\mu^\lambda M(w\cdot\mu), M(ww'\cdot\lambda)^\vee)\\ &= \dim\mathrm{Hom}_{\mathcal{O}}(M(w\cdot\mu), T_\lambda^\mu M(ww'\cdot\lambda)^\vee)\\ &= \dim\mathrm{Hom}_{\mathcal{O}}(M(w\cdot\mu), (T_\lambda^\mu M(ww'\cdot\lambda))^\vee)\\ &= \dim\mathrm{Hom}_{\mathcal{O}}(M(w\cdot\mu), M(ww'\cdot\mu)^\vee)\end{aligned}$$

The last line is 0 unless $ww'\cdot\mu = w\cdot\mu\Leftrightarrow w'\in W_\mu$, in which case it is 1. Also, $ww'\cdot\lambda = w\cdot\lambda\Leftrightarrow w'\in W_\lambda$, so the character equality follows. □

Writing each $M\in\mathcal{O}_\mu$ as $\mathbf{Z}$-linear combination of $M(w\cdot\mu)$, we conclude with the assumptions in the theorem that

$$\mathrm{ch}T_\lambda^\mu T_\mu^\lambda M = |W_\mu/W_\lambda|\mathrm{ch}M \tag{6.0.1}$$

Finally, we cite the following two general lemmas on category theory to deduce a nice corollary.

Lemma 6.0.1. [Gait, 4.27] *Suppose $F:\mathcal{C}_1\to\mathcal{C}_2$, $G:\mathcal{C}_2\to\mathcal{C}_1$ are mutually adjoint exact functors between Artinian abelian categories. Then they are mutually quasi-inverse equivalences if and only if they define mutually inverse isomorphisms on the level of Grothendieck groups.*

Given an abelian category $\mathcal{C}$ and a Serre subcategory $\mathcal{A}$ (so closed under extensions and subquotients), we may define the Serre quotient $\mathcal{C}/\mathcal{A}$ to be an abelian category equipped with an exact functor $\mathrm{quot}:\mathcal{C}\to\mathcal{C}/\mathcal{A}$ such that $\mathcal{A} = \mathrm{Ker}(\mathrm{quot})$ = full subcategory of $\mathcal{C}$ formed by objects A such that $\mathrm{quot}(A) = 0$.

Lemma 6.0.2. [BG, 2.4] *Let $F:\mathcal{C}_1\to\mathcal{C}_2$ be exact functor between abelian categories which has left adjoint F'. Then $\bar{F}:\mathcal{C}/KerF\to\mathcal{C}'$ is an equivalence of categories if and only if the canonical morphism $F\circ F'\to Id_{\mathcal{C}'}$ is an isomorphism.*

Conjecture: these lemma imply $\lambda \in F, \mu \in \overline{F} \Rightarrow T := T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ induces an equivalence of categories

$$\overline{T} : \mathcal{O}_\lambda / \mathrm{Ker} T \to \mathcal{O}_\mu,$$

where we know Ker T is Serre subcategory generated by $L(w \cdot \lambda)$ for all w except $w \cdot \mu \in \widehat{w \cdot F}$.

We conclude with a final theorem on translations from walls. Assume λ is regular, so $W_\lambda = \{1\}$ and $\lambda \in C$ lies in a chamber. Suppose μ lies in a single wall of C, the intersection $\overline{C} \cap H_\alpha$ for some $\alpha \in \Phi^+$. Write $s = s_\alpha$, so that $W_\mu = \{1, s\}$, and Equation 6.0.1 implies $\mathrm{ch} T_\lambda^\mu T_\mu^\lambda L(w \cdot \mu) = 2\mathrm{ch} L(w \cdot \mu)$. But there are no self extensions of simples, so

$$T_\lambda^\mu T_\mu^\lambda L(w \cdot \mu) \cong L(w \cdot \mu)^{\oplus 2}.$$

Next, consider $w \in W$ such that $w\alpha > 0 \Rightarrow w \cdot \mu \in \widehat{w \cdot C}$. By Theorem 5.2, we conclude $T_\lambda^\mu L(w \cdot \lambda) = L(w \cdot \mu)$ and $T_\lambda^\mu L(ws \cdot \lambda) = 0$.

Theorem 6.2. *Let λ be regular and lie in chamber C, and suppose μ lies in a single wall of C corresponding to $\alpha > 0$. Suppose $w \in W$ satisfies $w\alpha > 0$, so $\ell(ws) > \ell(w)$ with $s = s_\alpha$.*

(1) There is short exact sequence

$$0 \to M(ws \cdot \lambda) \to T_\mu^\lambda M(w \cdot \mu) \to M(w \cdot \lambda) \to 0$$

(2) Top$T_\mu^\lambda M(w \cdot \mu) = L(w \cdot \lambda)$, and the above sequence is nonsplit
(3) $T_\mu^\lambda L(w \cdot \mu)$ is self-dual with top and socle $L(w \cdot \lambda)$.
(4) $[T_\mu^\lambda L(w \cdot \mu) : L(w \cdot \lambda)] = 2$.
(5) $[T_\mu^\lambda L(w \cdot \mu) : L(ws \cdot \lambda)] = 1$.
(6) Suppose $w' \cdot \lambda \neq w \cdot \lambda$. If $[T_\mu^\lambda L(w \cdot \mu) : L(w' \cdot \lambda)] > 0$, then $w's \cdot \lambda < w' \cdot \lambda$ and $T_\lambda^\mu L(w' \cdot \lambda) = 0$.

Proof. (1) This follows from Theorem 6.1 and fact that $M(ws \cdot \lambda)$ occurs as submodule in our construction of standard filtration of translated Verma.
(2) Adjointness implies

$$\mathrm{Hom}_\mathcal{O}(T_\mu^\lambda M(w \cdot \mu), L(w' \cdot \lambda)) \cong \mathrm{Hom}_\mathcal{O}(M(w \cdot \mu), T_\lambda^\mu L(w' \cdot \lambda)$$

which by theorem 5.2 is nonzero precisely when $w' \cdot \mu = w \cdot \mu$ (so $w' = w$ or $w' = ws$) and $w' \cdot \mu$ lies in upper closure $\widehat{w' \cdot C}$. This forces $w' = w$.
(3) Exactness shows $T_\mu^\lambda L(w \cdot \mu)$ is quotient of $T_\mu^\lambda M(w \cdot \mu)$, and top maps to top. Hence (2) shows $L(w \cdot \lambda)$ is only simple quotient of $T_\mu^\lambda L(w \cdot \mu)$ and it appears with multiplicity 1. Moreover, $L(w \cdot \mu)$ is self dual under $\vee$, and translation functors commute with duality, so $T_\mu^\lambda L(w \cdot \mu)$ is self dual.
(4) From (3), $[T_\mu^\lambda L(w \cdot \mu) : L(w \cdot \lambda)] \geq 1$. Then we conclude from $[T_\lambda^\mu T_\mu^\lambda L(w \cdot \mu) : L(w \cdot \mu)] = 2$.
(5) From (1), $L(ws \cdot \lambda)$ has multiplicity 1 in $T_\mu^\lambda M(w \cdot \mu)$ and the module M generated by maximal vector of weight $ws \cdot \lambda$ being isomorphic to $M(ws \cdot \lambda)$. Thus $[T_\mu^\lambda L(w \cdot \mu) : L(ws \cdot \lambda)] \leq 1$. If M was in kernel of surjection $T_\mu^\lambda M(w \cdot \mu) \to T_\mu^\lambda L(w \cdot \mu)$, there would be just a single composition factor $L(w \cdot \lambda)$ in $T_\mu^\lambda L(w \cdot \mu)$, contradicting (4). So M has nonzero image, forcing $T_\mu^\lambda L(w \cdot \mu)$ to have composition factor $L(ws \cdot \lambda)$.
(6)If $w' = ws$, $w's \cdot \lambda - w \cdot \lambda < ws \cdot \lambda = w' \cdot \lambda$. So $T_\mu^\lambda L(w' \cdot \lambda) = 0$. In the case $w' \neq w, ws$, we have $L(w' \cdot \mu)$ not isomorphic to $L(w \cdot \mu)$. Then $[T_\mu^\lambda L(w \cdot \mu) : L(w' \cdot \lambda)] > 0 \Rightarrow [T_\lambda^\mu T_\mu^\lambda L(w \cdot \mu) : T_\lambda^\mu L(w' \cdot \lambda] > 0$. But $T_\lambda^\mu L(w' \cdot \lambda)$ is either 0 or $L(w' \cdot \mu)$ and $L(w' \cdot \mu)$ not isomorphic to $L(w \cdot \mu)$, so the first option holds. This then forces $w's \cdot \lambda < w' \cdot \lambda$ by 5.2. $\square$

References

[Gait] D. Gaitsgory, Geometric representation theory, lecture notes, Harvard Univ, 2005.
[BG] A. Beilinson, V. Ginzburg, Wall-crossing functors and $\mathcal{D}$-modules, Represent. Theory 3 (1999), 131.
[Hum08] J. Humphreys, Representations of Semisimple Lie Algebras in BGG Category O
[Jan] J.C Jantzen, Moduln mit einem holchsten Gewicht, Lecture Notes in Math., 750, Springer-Verlag, Berlin 1979.