Introduction to Soergel bimodules

1. Motivation

How to prove the Kazhdan-Lusztig conjecture for the characters of Vermas in the principal block $\mathcal{O}_0$?

$$[M(y\cdot 0) : L(x\cdot 0)] = \underset{\text{KL polynomials}}{h_{y,x}}(1)$$

We need to relate $\mathcal{O}_0$ and H, the Hecke algebra corresponding to (W,S). However $K_0(\mathcal{O}_0) = \mathbb{Z}[W]$ (no q).
Ideally we would like to relate $\mathcal{O}_0$ to some category of graded (bi)modules, whose Grothendieck group could be a $\mathbb{Z}[v,v^{-1}]$-algebra, hopefully H.

Idea of Soergel: study the functor $\mathbb{V} := \mathrm{Hom}_{\mathcal{O}_0}(P(w_0\cdot 0), -) : \mathcal{O}_0 \to \mathrm{End}(P(w_0\cdot 0))\text{-mod} = \mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}]^W_+\text{-mod}$

"coinvariant algebra" → naturally graded!

Now $\mathbb{V}$ is by no means an equivalence, but it is sufficient to prove the Kazhdan-Lusztig conjecture! In fact it suffices to show that the image of this category under $\mathbb{V}$ (the category of Soergel modules) categorifies the Hecke algebra in such a way that indecomposables categorify the KL basis. (More on this at the end). Then properties of this functor with our knowledge of category $\mathcal{O}$ give a clean proof of KL.

This categorification is an algebraic combinatoric endeavor and it will be our focus today.

2. The Hecke algebra

Definition. A **Coxeter system** (W,S) is a group and a finite set $S \subset W$ such that $W = \langle S \mid R \rangle$, where the set of relations is:

- $s^2 = 1$ $\quad \forall s \in S$ $\quad$ "quadratic"
- $\underbrace{sts\ldots}_{m_{st}} = \underbrace{tst\ldots t}_{m_{st}}$ $\quad \forall s,t \in S$ $\quad$ "braid"

Example: W = Weyl group, S = simple reflections

There is a faithful representation attached to every Coxeter system.

Definition (Geometric representation): Let $V = \mathbb{R}\{\alpha_s : s \in S\}$, where α_s are formal variables. Define a form $(-,-)$ on V by

$$(\alpha_s, \alpha_t) = -\cos \frac{\pi}{m_{st}}$$

Then S acts on V via $\lambda \mapsto s(\lambda) := \lambda - 2(\lambda, \alpha_s)\alpha_s$

(If (W,S) comes from a Lie algebra, this is the "tautological" representation of the Weyl group).

Definition. The Hecke algebra associated to (W,S) is the unital associative algebra $H = H(W)$ over $\mathbb{Z}[v,v^{-1}]$ generated by the symbols $\{\delta_s : s\in S\}$ such that

- $\delta_s^2 = (v^{-1}-v)\delta_s + 1$ "quadratic" $\leftrightarrow (\delta_s - v^{-1})(\delta_s+v) = 0$ "eigenvalues of δ_s are v^{-1} and $-v$"
- $\underbrace{\delta_s\delta_t\cdots}_{m_{st}} = \underbrace{\delta_t\delta_s\cdots}_{m_{st}}$ "braid"

Note that specializing to $v=1$ one gets the group algebra $\mathbb{C}[W]$

Two $\mathbb{Z}[v,v^{-1}]$-bases:

- Standard: $\{\delta_x : x\in W\}$ (Here we take a reduced expression $\underline{x} = s_1 \cdots s_n$ and define $\delta_x = \delta_{s_1}\cdots\delta_{s_n}$
- Kazhdan-Lusztig: $\{b_x : x\in W\}$ characterized by: $\begin{cases} \overline{b_x} = b_x & \text{(KL involution: } \overline{\delta_s} := \delta_s^{-1} = \delta_s + (v-v^{-1}),\ \overline{v} := v^{-1} \text{ extended multiplicatively)} \\ b_x = \delta_x + \sum_{y<x} h_{yx}\delta_y & \text{for some } h_{yx}\in v\mathbb{Z}[v] \text{ "degree bound"} \end{cases}$

In particular, $b_s = \delta_s + v$, so that
$\overline{b_s} = \delta_s + (v-v^{-1}) + v^{-1} = b_s$

Example $W = S_3$, $S = \{s,t\}$

Easy fact: $\delta_x\delta_s = \begin{cases} \delta_{xs} & xs>x \\ \delta_{xs} + (v^{-1}-v)\delta_x & x<xs \end{cases}$

$b_1 = 1$

$b_s = \delta_s + v$

$b_t = \delta_t + v$

$\underbrace{b_t b_s}_{\text{self-dual}} = \underbrace{\underbrace{\delta_t\delta_s}_{\delta_{ts}} + v\delta_t + v\delta_s + v^2}_{\text{degree bound}}$ $\rightsquigarrow$ $b_{ts} = b_t b_s$

$b_{st} = b_s b_t$

$b_{ts}\cdot b_t = (\delta_{ts} + v\delta_t + v\delta_s + v^2)(\delta_t + v)$

$= \underbrace{\delta_{ts}\delta_t}_{\delta_{tst}} + v((v^{-1}-v)\delta_t + 1) + v\delta_{st} + v^2\delta_t + v\delta_{ts} + v^2\delta_t + v^2\delta_s + v^3$

$= \delta_{tst} + v\delta_{st} + v\delta_{ts} + v^2\delta_s + (1+v^2)\delta_t + (v+v^3)$

$\Rightarrow b_{tst} = b_{ts}\cdot b_t - b_t$

3. Soergel bimodules

Let V be the geometric representation of V, and let $R = \mathrm{Sym}(V)$. We view this as a graded algebra where $\deg(V) = 2$. We can also write $V = \mathbb{R}[\alpha_s : s \in S]$ and set $\deg(\alpha_s) = 2$. (Real version of $\mathcal{U}(\mathfrak{h}^*)\ldots$)
Note that we have a natural action $W \circlearrowright R$.

Take a set $I \subset S$ (such that $W_I := \langle I \rangle$ is finite). We will use the notation $R^I = R^{W_I}$ (sometimes dropping brackets)

For instance, if $W = S_2$, $S = \{s\}$. Then $R = \mathbb{R}[\alpha]$, $s(\alpha) = -\alpha$ and $R^s = \mathbb{R}[\alpha^2]$. In fact we have that as R^s-bimodules, $R = R^s \oplus R^s\alpha$. Since α_s has degree 2, we have an isomorphism of graded R^s-bimodules

For me, $M(i)^j = M^{i+j}$

$$R = R^s \oplus R^s(-2).$$

This holds for any (W,S) after fixing $s \in S$. In this case $R^s = \mathbb{R}[\alpha_s^2, \alpha_t + \cos(\frac{\pi}{m_{st}})\alpha_s : t \neq s]$

$s(\alpha_t + \cos(\frac{\pi}{m_{st}})\alpha_s) = \alpha_t + 2\cos(\frac{\pi}{m_{st}})\alpha_s - \cos(\frac{\pi}{m_{st}})\alpha_s$

We still have a map $\partial_s : R \to R^s(-2)$ such that $0 \to R^s \to R \to R^s(-2) \to 0$ splits.

<u>Definition</u>. For $s \in S$ the Demazure operator $\partial_s : R \to R^s(-2)$ is the graded map $f \mapsto \frac{f - s(f)}{\alpha_s}$ $\left(\alpha_s \mid \underbrace{f - s(f)}_{\text{antiinv.}}\right)$

Then we have an isomorphism of R^s-bimodules
$$\begin{aligned} R &\to R^s \oplus R^s(-2) \\ f &\mapsto (\partial_s(f\alpha_s), \partial_s(f)) \end{aligned}$$
(Inverse: $(g,h) \mapsto \frac{1}{2}g + \frac{1}{2}\alpha_s h$)

<u>Lemma</u> (properties of Demazure operators)

(prove only:)

(1) ∂_s is an R^s-bimodule map.
(2) $s \circ \partial_s = \partial_s$, $\partial_s \circ s = -\partial_s$
(3) $\partial_s^2 = 0$
(4) Twisted Leibniz rule: $\partial_s(fg) = \partial_s(f)g + s(f)\cdot\partial_s(g)$
(5) $(f,g)_s := \partial_s(fg)$ is a perfect pairing $R \times R \to R^s$
(6) $\underbrace{\partial_s\partial_t\ldots}_{m_{st}} = \underbrace{\partial_t\partial_s\ldots}_{m_{st}}$

<u>Proof</u>: (1) For $f \in R^s$, $g \in R$, $\partial_s(fg) = \frac{fg - s(fg)}{\alpha_s} = \frac{fg - fs(g)}{\alpha_s} = f\partial_s(g)$.

(2) For $f \in R^s$, $s(\partial_s(f)) = \frac{s(f) - f}{-\alpha_s} = \partial_s(f)$. Also, $\partial_s(s(f)) = \frac{s(f) - f}{\alpha_s} = -\partial_s(f)$.

(3) $\partial_s|_{R^s} = 0$ and $\mathrm{Im}(\partial_s) = R^s(-2)$.

(4) $\partial_s(fg) = \frac{fg - s(fg)}{\alpha_s} = \frac{fg - s(f)g}{\alpha_s} + \frac{s(f)g - s(f)s(g)}{\alpha_s} = \partial_s(f)g + s(f)\partial_s(g)$.

(5) $R \xrightarrow{\sim} \mathrm{Hom}_{R^s}(R, R^s)$?

$f \mapsto (g \mapsto \partial_s(fg))$

Injective: $\partial_s(fg) = 0 \ \forall g \Rightarrow \partial_s(f) = \partial_s(\alpha_s f) = 0 \Rightarrow f = 0$

Surjective: let $\varphi : R \to R^s$, $1 \mapsto f$, $\alpha_s \mapsto g$. Then the map $h \mapsto \partial_s((\frac{\alpha_s}{2} f + \frac{1}{2} g)h)$ sends $1 \mapsto \partial_s(\frac{\alpha_s}{2} f + \frac{1}{2} g) = f$, $\alpha_s \mapsto \partial_s(\frac{1}{2} f + \frac{\alpha_s}{2} g) = g$

(6) $m_{st} = 2 \Rightarrow st = ts \Rightarrow \partial_s \partial_t (f) = \partial_s\left(\frac{f - t(f)}{\alpha_t}\right) = \dfrac{\frac{f - t(f)}{\alpha_t} - \frac{s(f) - st(f)}{\alpha_t}}{\alpha_s} = \dfrac{f - t(f) - s(f) + st(f)}{\alpha_t \alpha_s} = \partial_t \partial_s (f)$

$m_{st} = 3 \ldots$

<u>Definition</u>: $\partial_w := \partial_{s_1} \cdots \partial_{s_n}$, for $\underline{x} = s_1 \cdots s_n$ a reduced expression.

<u>Remark</u>: $\{\partial_w\}_{w \in W}$ are a basis for the NilCoxeter algebra associated to (W,S) (by definition)

We restrict our attention to the monoidal category of graded R-bimodules which are f.g. as left and right R-modules.
We will sometimes write $\cdot$ for $\otimes_R$.

<u>Definition</u> Denote $B_s := R \otimes_{R^s} R(1)$. The **Bott-Samelson bimodule** corresponding to an expression $\underline{w} = s_1 \cdots s_n$ is $BS(\underline{w}) := B_{s_1} \cdots B_{s_n}$

<u>Remarks</u>:
- $BS(\underline{w}) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R\,(\ell(\underline{w}))$
- The element $1 \otimes_{R^{s_1}} \cdots \otimes_{R^{s_n}} 1$ lives in degree $-\ell(\underline{w})$.
- $BS(\underline{u}) \cdot BS(\underline{v}) = BS(\underline{uv})$
- (*) $B_s = R \otimes_{R^s} (R^s \oplus R^s(-2))(1) = R \otimes_{R^s} R^s(1) \oplus R \otimes_{R^s} R^s(-1) \underset{\text{as graded left } R\text{-modules}}{=} R(1) \oplus R(-1)$. (substituting on the left tensor gives the result for right R-modules)

<u>Proposition</u>. Any BS bimodule is graded free as a left (right) R-module.

<u>Proof</u>: they are tensor products of free R-modules (see remark). □

Q: what is the splitting in (*)?

A: $B_s = \underbrace{R \cdot \underbrace{(1 \otimes 1)}_{\deg -1}}_{R(1)} \oplus \underbrace{R \cdot \underbrace{\tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)}_{\deg +1}}_{R(-1)}$

Some geometric motivation: let P_i be the minimal parabolic corresponding to the simple root α_i. Then B^k acts on $P_1 \times \cdots \times P_k$ via $(p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \ldots, b_{k-1} p_k b_k^{-1})$
and $P_1 \times \cdots \times P_n / B^k = BS_{\underline{w}}$, the Bott-Samelson variety. We have an obvious map $BS_{\underline{w}} \to G/B$, whose image is the Schubert variety $\overline{BwB}/B$, and this map is a resolution of singularities. Now $B \circlearrowright BS_{\underline{w}}$ and $BS(\underline{w}) = H^*_B(BS_{\underline{w}})$.

<u>Definition</u>. A **Soergel bimodule** is a direct summand of a finite direct sum of grading shifts of Bott-Samelson bimodules.
The category of Soergel bimodules with morphisms given by maps of graded bimodules is denoted **SBim**.

Alternatively: the $(\oplus, \overset{\oplus}{\subseteq}, \otimes, (n))$-category generated by R, $B_s \ \forall s \in S$.
Warning: morphisms in SBim have degree 0, but in BSBim they are allowed to have any degree... (omit)
Fact: SBim has the Krull-Schmidt property: every object decomposes uniquely into finitely many indecomposables.

Break?

4. Categorification

So how does SBim categorify H? Let us look at indecomposables in some examples.

Example: $W = S_2$, $S = \{s\}$. Then R is indecomposable. How about B_s?

Lemma: If a graded R-bimodule is generated by a single homogeneous element (i.e. $M = RmR$), then M is indecomposable.

Proof: Let $d = \deg(m)$. Then $M^d = (RmR)^d = R^0 m R^0 = \mathbb{R}m$. Now suppose $M = L \oplus D$. Then $M^d = L^d \oplus D^d$, so m lies in one of the two and the other one must be zero. $\square$

We can conclude that $B_s = R(1 \otimes_{R^s} 1)R$ is indecomposable.

More indecomposables? Look at

$$\begin{aligned} B_s B_s &= R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &= R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \\ &= R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R \\ &= B_s(1) \oplus B_s(-1) \end{aligned}$$

Clearly R and B_s are the only indecomposables up to shift. Spoiler: in the Hecke algebra, $b_s^2 = (v + v^{-1}) b_s$.

Example: $W = S_3$, $S = \{s, t\}$.

So far we have R, B_s, B_t. Now $B_s B_t = R \otimes_{R^s} R \otimes_{R^t} R(2)$.

Claim: $B_s B_t$ is generated by $1 \otimes_{R^s} 1 \otimes_{R^t} 1$, and is therefore indecomposable. [This trick will work for any pair s,t with $m_{st} \neq \infty$]

Proof: It suffices to show that the middle R is generated, in other words, that $R^s 1 R^t = R^s + R^t = R$.
Recall that $R^s = \mathbb{R}[\alpha_s^2, \alpha_s + \frac{1}{2}\alpha_t]$, $R^t = \mathbb{R}[\alpha_t^2, \alpha_t + \frac{1}{2}\alpha_s]$. $\square$ (This will hold whenever $m_{st} \neq \infty$)

We denote $B_{st} = B_s \cdot B_t$, $B_{ts} = B_t \cdot B_s$. Observe that this proves $B_s \not\cong B_t$ since B_s^2 is decomposable while $B_s B_t$ is not. (recall $b_s b_t = b_{st}$)

The next case to look at is $B_s B_t B_s = R \otimes_s R \otimes_t R \otimes_s R(3)$. It has a copy of $R \otimes_{s,t} R(3)$: $f \otimes_{s,t} g \overset{\varphi}{\mapsto} f \otimes_s 1 \otimes_t 1 \otimes_s g$ though it is not clear that $\mathrm{Im}(\varphi)$ is in SBim.

$c_t \cdot f = f \cdot c_t$ ...

We also have a map $\psi : B_s \longrightarrow B_s B_t B_s$

$$1 \otimes_s 1 \longmapsto \frac{1}{2}\left(1 \otimes_s \alpha_t \otimes_t 1 \otimes_s 1 + 1 \otimes_s 1 \otimes_t \alpha_t \otimes_s 1\right) = 1 \otimes_s \Big(\underbrace{\tfrac{1}{2}(\alpha_t \otimes_t 1 + 1 \otimes_t \alpha_t)}_{c_t}\Big) \otimes_s 1$$

Clearly $\mathrm{Im}(\varphi) \cap \mathrm{Im}(\psi) = 0$, and some computation shows $\mathrm{Im}(\varphi) + \mathrm{Im}(\psi) = R \otimes_s R \otimes_t R \otimes_s R(3)$.

It follows that $B_{sts} := R \otimes_{s,t} R(3)$ lies in SBim since $B_s B_t B_s = B_s \oplus B_{sts}$ (compare to $b_{sts} = b_s b_t b_s - b_s$)

Note that what we would call B_{tst} is again $R\otimes_{R^{s,t}} R$, so "$B_{sts}\cong B_{tst}$".

Next, observe that $B_sB_tB_s \simeq B_{sts}\oplus B_s$

$$B_sB_sB_t \simeq B_sB_t(-1)\oplus B_sB_t(1)$$

It follows that $B_{st}\neq B_{ts}$!

Finally, recall that we have a splitting $R = R_s\oplus R_s(-2)$, which can be regarded as one of $(R^{s,t},R^s)$-bims.

Therefore $B_{sts}B_s = R\otimes_{R^{s,t}} R\otimes_{R^s} R(4)$

$$= R\otimes_{R^{s,t}}(R_s\oplus R_s(-2))\otimes_{R^s} R(4)$$

$$= R\otimes_{R^{s,t}} R(4)\oplus R\otimes_{R^{s,t}} R(2) = B_{sts}(1)\oplus B_{sts}(-1).$$

Therefore $B_{sts}B_s = B_{sts}(1)\oplus B_{sts}(-1) = B_{tst}B_t = B_{sts}B_t$

$$B_sB_{sts} \underset{\substack{\downarrow\\ \text{same calculation, other side}}}{=} B_{sts}(1)\oplus B_{sts}(-1) = B_tB_{tst} = B_tB_{sts}$$

This shows:

- There are no more indecomposables
- $B_{sts}\not\cong$ the previous ones

In summary, there is an indecomposable per element of W, and equalities on the level of H are lifted to isomorphisms in $\mathbb{S}\mathrm{Bim}$.

Example: $W = S_2\times S_2$, $S=\{s,t\}$

Then $R\otimes_{R^{s,t}}R(2)\xrightarrow{\sim} B_sB_t = R\otimes_{R^s}R\otimes_{R^t}R(2)$

$$1\otimes_{s,t}1 \longmapsto 1\otimes_s 1\otimes_t 1$$

since $R^{s,t} = \mathbb{R}[\alpha_s^2,\alpha_t^2]$ and so given $f\otimes_s g\otimes_t h$, we can write $g = \underset{\substack{\uparrow\\ R^{s,t}}}{g_0}+\alpha_t g_1+\alpha_s g_2$ uniquely

i.e. $f\otimes_s g\otimes_t h = (f+g_0+\alpha_t g_1)\otimes_s 1\otimes_t(h+\alpha_s g_2)$.

This gives a well defined map (by uniqueness) which is inverse to the previous one. It follows that $B_{st}=B_{ts}$, and therefore $B_{st}B_s = B_{ts}B_s \to$ no new indecomp. $\Rightarrow \{R, B_s, B_t, B_{st}\}$

5. Upshot: Soergel's categorification theorem

Consider the split Grothendieck group of SBim, i.e. the abelian group generated by the symbols $[B]$ for $B \in Ob(SBim)$, subject to $[B] = [B'] + [B'']$ whenever $B = B' \oplus B''$. This has a ring structure due to the monoidal structure on SBim, and grading shifts make it into a $\mathbb{Z}[v, v^{-1}]$-algebra via $v \cdot [B] = [B(1)]$.

<u>Soergel's categorification theorem</u>

(1) There is a bijection $W \longleftrightarrow \{\text{indecomposables in } SBim\}/\text{shift}$

$$w \longmapsto B_w$$

For a reduced expression $w = s_1 \cdots s_n$, B_w is a summand of $B_{s_1} \cdots B_{s_n}$

(2) There is an isomorphism

$$H \longrightarrow [SBim]_\oplus$$

$$b_x \longmapsto [B_x]$$

with inverse $ch : SBim \to H$ (ch can be defined explicitly)

Remark: the fact that $ch(B_x) = b_x$ was known as Soergel's conjecture. Soergel proved it appealing to the decomposition theorem from geometry. An algebraic proof for all Coxeter systems was published in 2014 by Elias + Williamson. Incidentally, this proves the positivity conjecture (for all Coxeter systems).

Final remark: in this talk we essentially proved that the map $H \to [SBim]_\oplus$ is a homomorphism if $m_{st} \in \{2, 3\}$ $\forall s,t \in S$. However, proving this in general manipulating polynomials becomes very difficult and motivates defining diagrammatics for this category.