Classical Theory of SSIM 1

Thursday, February 18, 2021 7:54 PM

Not ${}_R M_S$ means M is a (R,S) bimodule

Def $\mathcal{B}im$ is the 2-cat
- obj: rings R (cat R-mod)
- 1-morph: $S \xrightarrow{F_M} R \longleftrightarrow {}_R M_S$
  (S-mod $\xrightarrow{M \otimes_S -}$ R-mod)
- 2-morph: $F_M \to F_N \longleftrightarrow g: M \to N$ (bimodule map)
  (natural transformation $M \otimes_S - \to N \otimes_S -$)

Def Given (W,S), $I \subset S$ is called finitary if W_I is finite. Let

$$w_I := \text{longest ele of } W_I$$

$$\ell(I) := \ell(w_I)$$

Let V be a reflection faithful rep of W $\begin{pmatrix} V_{geo} - W \text{ finite} \\ V_{KM} - W \text{ affine} \end{pmatrix}$

Let $R = \mathrm{Sym}\, V^*$

Def Singular Bott-Samelson bimodules ($\mathbb{S}\mathbb{B}\mathbb{S}\mathcal{B}im$) is the full sub 2-cat of $\mathcal{B}im$
- obj: finitary subsets $I \subset S$ (R^I)
- 1-morph: gen by $\otimes$ of

$${}_J R^J_I \ , \ {}_I R^J_J(\ell(I) - \ell(J))$$

$$\left(\begin{array}{l} \text{composition of functors} \\ \mathrm{Ind}^{R^J}_{R^I} = R^J \otimes_{R^I} (-) \\ g\mathrm{Res}^{R^J}_{R^I} = R^J(\ell(I) - \ell(J)) \otimes_{R^J} (-) \end{array}\right)$$

when $R^I \subset R^J$, i.e. $J \subset I$
- 2-morph: bimodule maps b/t

Rem 1-morph (bimodules) of $\mathbb{S}\mathbb{B}\mathbb{S}\mathrm{Bim}$ correspond to sequences

$$(I_1, I_2, \ldots, I_d) \quad I_k \subset S \text{ finitary}$$

where we read right to left, and
(1) Start out with R^{I_d}
(2) As we read right to left
- if $I_k \subset I_{k+1}$, apply $\mathrm{Ind}_{I_{k+1}}^{I_k}$
- if $I_k \supset I_{k+1}$ apply $g\mathrm{Res}_{I_k}^{I_{k+1}}$

$$\left(M \longmapsto {}_{I_k}M(\ell(I_k) - \ell(I_{k+1})) \right)$$

Ex 1: $(\phi, \{s_1\}, \phi)$

$$M = \mathrm{Ind}_{s_1}^{\phi} \circ g\mathrm{Res}_{s_1}^{\phi}(R^{\phi}) = R \otimes_{R^{s_1}} R(1-0)$$

$$= R \otimes_{R^{s_1}} R(1) = B_s$$

Ex 2: $(\phi, s_1, \ldots, \phi, s_d, \phi) = BS(s_1, \ldots, s_d)$

$$= R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_d}} R(d)$$

Def Singular Soergel Bimodules is 2-cat $\mathbb{S}\mathbb{S}\mathrm{Bim}$ = graded Karoubian envelope of $\mathbb{S}\mathbb{B}\mathbb{S}\mathrm{im} = \langle \mathbb{S}\mathbb{B}\mathbb{S}\mathrm{Bim} \rangle_{\oplus, (1), \oplus_c}$

Ex 3: $(\phi, \{s,t\}, \phi)$

$$= R \otimes_{R^{s,t}} R(m_{st}) = B_{\underline{st\ldots s}}$$

m_{st}

- this gave an indecomp Soergel Bimod not just a B-S bimodule!

Rem; B/c $B_s \in S\mathbb{S}\mathrm{Bim}(\emptyset,\emptyset)$

$\Rightarrow \mathbb{S}\mathrm{Bim} \subseteq S\mathbb{S}\mathrm{Bim}(\emptyset,\emptyset)$

Thrm We have an equivalence of cats

$$\mathbb{S}\mathrm{Bim} \simeq S\mathbb{S}\mathrm{Bim}(\emptyset,\emptyset)$$

as subcat of (R,R) bimodules

This is not obvious at all, aka why is

$$R \otimes_{R^{s,t,u}} R^s \otimes_{R^{s,u}} R \in \mathbb{S}\mathrm{Bim}\ ?$$

- Recall Soergel's Cat Thrm

(1) There is bijection $W \longleftrightarrow \{\text{indecomp in } \mathbb{S}\mathrm{Bim}\}$ (1)

(2) There is iso $H \xrightarrow{\sim} [\mathbb{S}\mathrm{Bim}]_\oplus$

Soergel-Williamson Cat Thrm;

(1) There is a bijection $W_J \backslash W / W_I \longleftrightarrow \{\text{indecomp in } S\mathbb{S}\mathrm{Bim}(I,J)\}$

(2) There is an equivalence of cat

$$\mathcal{H}(W) \simeq [S\mathbb{S}\mathrm{Bim}]_\oplus$$

Hecke algebroid ← k-linear cat w/ obj I

Ex 1: $\mathrm{Hom}_{\mathcal{H}}(\emptyset,\emptyset) = H$

Ex 2: $\mathrm{Hom}_{\mathcal{H}}(\emptyset, I) = H \otimes_{H_I} \mathrm{triv}$ ← parabolic KL

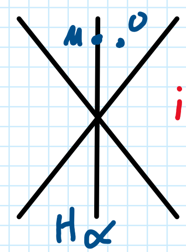
Q: What about $S\mathbb{S}\mathrm{Bim}$ is singular?

A: $\mathbb{S}\mathrm{Bim} \simeq S\mathbb{S}\mathrm{Bim}(\emptyset,\emptyset) \xrightarrow{\otimes_k k} \mathbb{S}\text{-mod} \simeq \mathrm{Proj}\,\mathcal{O}_0$

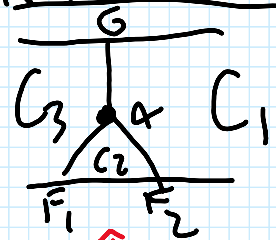
$S\mathbb{S}\mathrm{Bim}(\emptyset, I) \xrightarrow{\otimes_k k} S\mathbb{S}\text{-mod} \simeq \mathrm{Proj}\,\mathcal{O}_\mu$ ← is a sing block of $\mathcal{O}$

μ is on the I-wall

when W is a Weyl group



Recall string diagrams for 2-cat



- regions = obj
- lines = 1-morph
- dots = 2-morph

2-morph

$\alpha : F_1 \circ_h F_2 \longrightarrow G$, $\quad F_1 \circ F_2 : C_1 \to C_3$ 1-mor, $\quad G : C_1 \to C_3$

- Read right to left, bottom to top

Singular Diagrammatics for Frob ext

- Let $i : A \hookrightarrow B$ be a Frobenius extension

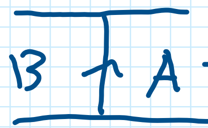
$\partial : B \to A$ be trace

$m : B \otimes_A B \to B$ mult

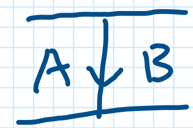
$\Delta : B \to B \otimes_A B$ coprod, $\Delta(1) = \sum_i b_i \otimes b_i^*$

where $\partial(b_i b_j^*) = \delta_{ij}$, aka dual basis

Generators: In 2-cat Bim

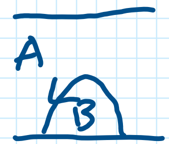


$= \mathrm{Ind}_A^B$

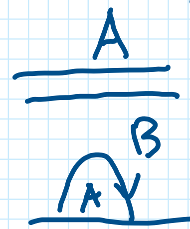


$= \mathrm{Res}_A^B$

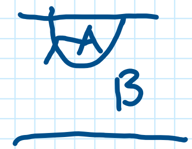
$= i$



$= \partial$



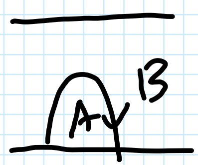
$= m$



$= \Delta$

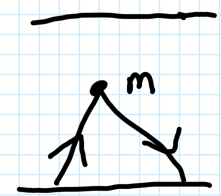
RHR: smaller ring is to the right of ↑

Check diagrams make sense



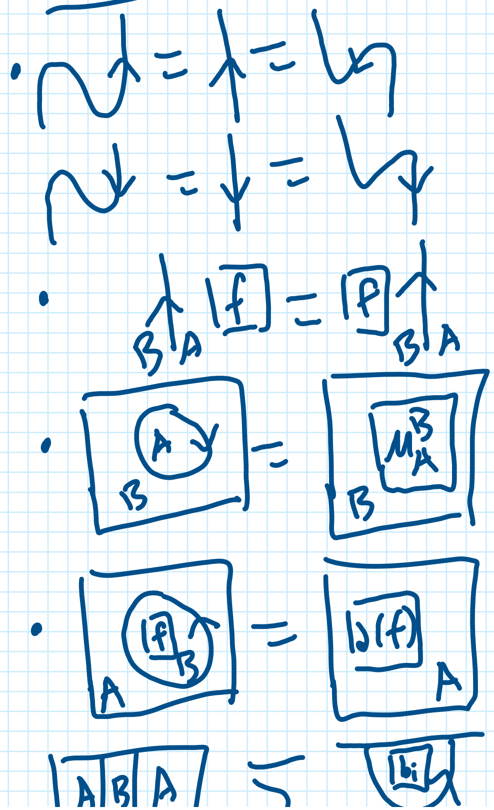
$= \begin{array}{c} B \\ \uparrow \\ B \otimes_A B \end{array}$

Technically should draw



But i, ∂, m, Δ will be only 2-morph b/t each of their respective 1-morph we will consider so omit dot

Also need 2-morph gen $\boxed{f}\ A$ (mult by $f \in A$) $\boxed{g}\ B$ (mult by $g \in B$)

Relations: Diagram	Corresponding Algebra
	• $\mathrm{Ind}_A^B \dashv\vdash \mathrm{Res}_A^B$ is biadjoint for Frob ext • i is A-mod map • $\mu_A^B = \sum b_i b_i^*$. LHS is literally $\mu \circ \Delta$ Barbell • LHS is literally $\partial(f\, i(1))$ keyhole • ${}_A B_A$ $\mathrm{id}\uparrow = \sum i(\partial(b_i^* -))\, b_i$

Thm: The 2-cat w/ these generators and relations denoted $\mathrm{Frob}(A \subset B) \simeq$ full sub 2-cat of Bim generated by • Ind_A^B
• Res_A^B

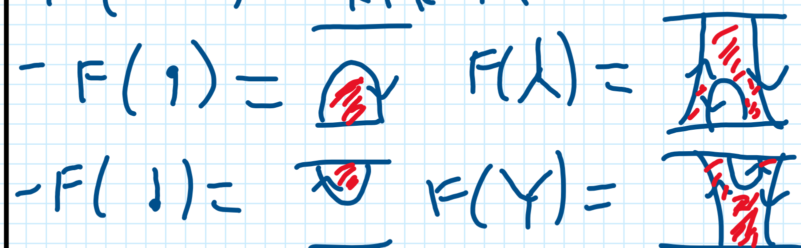
Relation to 1-color Soergel calculus

- Consider Frob ext $A = R^s \hookrightarrow R = B$

Thm: Let (W,S) be type A_1. There is a monoidal equivalence of cat

$$F: \mathcal{H}_{BS} \longrightarrow \mathrm{Frob}(R^s \hookrightarrow R)$$

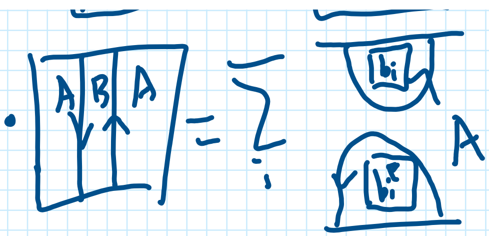
- $F(\text{—•—}) = {}_R\uparrow_{R^s}\downarrow_R =: \uparrow\ \downarrow$ R^s region



Soergel calculus $\longleftarrow$ Singular Soergel cal diagrams

"deformation retract" $\longleftarrow$ to R^s/red regions

Exercise 8.34 Deduce the following relation.

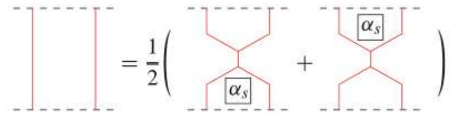


$\mathrm{id}_{{}_A B_A} = \sum_i \partial(b_i^* -)\, b_i$

aka $\{b_i\}, \{b_i^*\}$ are dual bases

neck-cutting

Exercise 8.34 Deduce the following relation.



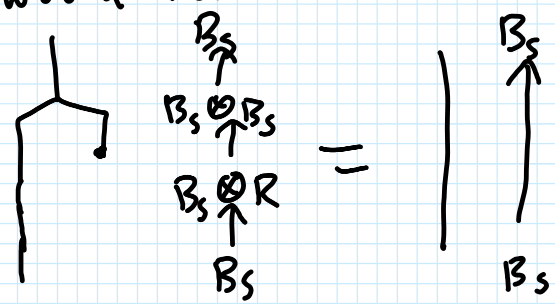
(8.27)

$\left(\mu^{R}_{R^s} = \frac{1}{2}\alpha_s + \frac{1}{2}\alpha_s = \alpha_s \right)$

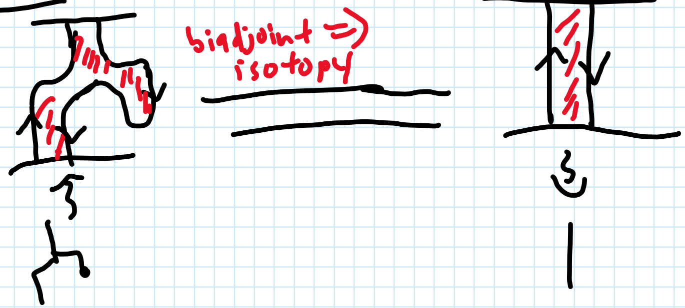
Why use SSC over SC?

1. Some relations become obvious in SSC

Ex: Before we claimed B_s was a Frob alg object in R-bimod. In order to check that diagramatics matches the alg, one would need to check rel such as



Now: Fatten LHS



Before: Check all axioms of Frob alg object ⇒ isotopy

Now: Isotopy one level up ⇒ Frob alg obj axioms

2. Explain why $JW_{\underline{w}}$ picks out the indecomposable summand B_w in $\mathcal{H}_{diag}$

Recall we defined a diagramatic morphism $JW_{(s,t,\dots)}$ in $\mathcal{H}_{diag}(s,t)$

1. Defining JW_n in $TL_n(\delta)$
2. Considering JW_n in $2TL_n(\delta)$ by coloring regions alternating colors, and set $\delta = a_{st} = \partial_s(\alpha_t)$
3. Apply "deformation retract"

Q: Why go through $2TL_n(\delta)$ to define?

Q: Why does this procedure give me a projector to B_w (indecomp SB)?

Big Picture of the role of SSBim in Representation Theory 1

- Let $\mathfrak{g}$ be a s.s. lie algebra w/ root datum $(P, Q, P^\vee, Q^\vee)$
- Have equivalence of cat

f.d. rep $\quad \mathrm{Rep}_f\,\mathfrak{g} \longleftrightarrow \mathrm{Rep}\, G_{sc}$

- G_{sc} = s.c., connected alg group, $\mathrm{Lie}(G_{sc}) = \mathfrak{g}$

$\Rightarrow$ $\mathrm{Rep}_f\,\mathfrak{g}$ splits into subcat by "central char"

$$\mathrm{Rep}_f\,\mathfrak{g} = \bigoplus_{z\in\hat\Omega\simeq\Omega} (\mathrm{Rep}_f\,\mathfrak{g})_z$$

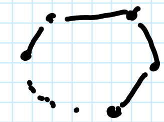
Ω = center of G_{sc} = P/Q ← a finite group!

$\Rightarrow$ Can make $\mathrm{Rep}_f\,\mathfrak{g}$ into a 2-cat $R_\mathfrak{g} = \bigoplus_{z,z'\in\Omega} {}_zR_{z'}$

- obj: elements of Ω
- 1-morph: ${}_zR_{z'} = (\mathrm{Rep}_f\,\mathfrak{g})_z \xrightarrow{-\otimes V} (\mathrm{Rep}_f\,\mathfrak{g})_{z'}$
- 2-morph: $\mathfrak{g}$-mod morph

- Let $W_a^\vee = W \ltimes Q$, $S_a^\vee = S \cup \{s_0\}$

$\Omega \curvearrowright (W_a^\vee, S_a^\vee)$,
- Every facet of alcoves $\leftrightarrow$ some de $S_a^\vee$
- translation by P preserves alcoves

Ex: $\widetilde{A_{n-1}}$ $\quad \Omega = \mathbb{Z}/n\mathbb{Z} \curvearrowright$  by rotations

Thm (Satake)

$$[R_\mathfrak{g}]\otimes\mathbb{Z}[v^{\pm1}] \simeq \bigoplus_{z,z'\in\Omega} \mathrm{Hom}_{\mathcal{H}(W)}(z(S), z(S'))$$

← Hecke algebroid

class of simples $\longleftrightarrow$ KL bases

Can we categorify this?

Conjecture (Soergel Satake equivalence)

There is an equivalence of 2-cats

$$R_\mathfrak{g} = \bigoplus_{z,z'\in\Omega} {}_zR_{z'} \simeq \bigoplus_{z,z'\in\Omega} \mathrm{SSBim}^{W_a^\vee}_{V_{KM}}(z(S), z'(S))$$

Conjecture (Alg Satake equivalence)

There is an equivalence of 2-cats

$$\mathrm{Fund}^{\Omega}_{(q)}(\mathfrak{sl}_n) \simeq \mathrm{mSBSBim}_{(q)}(W_a^\vee, V_{KM})$$

$\oplus_c$ $\quad \oplus_c$

- 1-morph for LHS now $\otimes$ of fund rep
- basis for Hom of LHS given by $\mathfrak{sl}_n$-webs by [CKM]
- proved for $\mathfrak{sl}_2, \mathfrak{sl}_3$, $n \geq 4$ don't have diagramatics

Enlightening example ($\mathfrak{sl}_2$) $\mathrm{Fund}^{\Omega}_{:q:}(\mathfrak{sl}_2)$

- $\mathfrak{g} = \mathfrak{sl}_2$, $W_a^\vee(\mathfrak{sl}_2) = \langle s,t \mid s^2 = t^2 = id \rangle$, $\Omega = \{\pm I\}$

$\Rightarrow \mathrm{Fund}^{\Omega}_q(\mathfrak{sl}_2) = \mathrm{Fund}_+ \oplus \mathrm{Fund}_-$ ← where $-I$ acts by -1

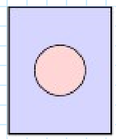
- obj: $\{+,-\}$
- 1-morph: $\mathfrak{sl}_2$ has only 1-fund rep $V = \mathbb{C}^2$ and it's odd $\Rightarrow$ 1-morph gen by $V \otimes -$ b/t obj

$\Rightarrow$ 1-morph gen by $+|-$, $-|+$

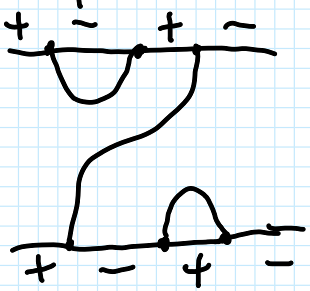
- 2-morph: intertwiners b/t $V^{\otimes n} \to V^{\otimes m}$

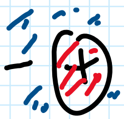
Thm: (1) $\mathrm{Fund}_{:q:}(\mathfrak{sl}_2) \simeq TL(-[2]_{:q:})$

(2) $\mathrm{Fund}^{\Omega}_q(\mathfrak{sl}_2) \simeq 2TL(-[2]_{:q:})$



Ex:



rel  $= -[2]_q$

 $= -[2]_q$

- what are a_{ts}, a_{st} for V_{km}? a_{ts}:

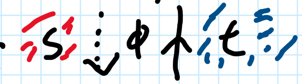
$m\mathbb{S}\mathbb{S}\mathrm{Bim}_{:q:}(W_a^\vee(\mathfrak{sl}_2), V_{km})$ $S = \{s,t\}$

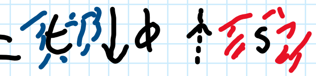
- obj: $\{s,t\}$
- 1-morph: $\mathbb{S}\mathbb{S}\mathrm{Bim}(s,t) \oplus \mathbb{S}\mathbb{S}\mathrm{Bim}(t,s)$

- Technically haven't discussed 2-colors yet

- However, notice $|\langle s \rangle \backslash \langle s,t \rangle / \langle t \rangle| = 1$

S-WCT $\Longrightarrow$ both cat above only have 1 indecomp

- As ${}_{R^s}R_{R^t}(1) :=$ 

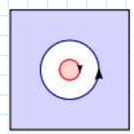
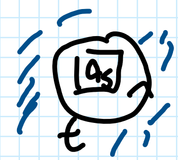
${}_{R^t}R_{R^s}(1) :=$ 

are both indecomposable (is gen by 1 as a (R^s, R^t) bimod as $R^s + R^t = R$), 1-morph are generated by these bimodules/diagrams

2-morph: By above suffice to just consider relations for the separate Frob ext $R^s \hookrightarrow R$, $R^t \hookrightarrow R$

$\Rightarrow$ gen by , etc

Most important relation:

 $=$  $= \partial_t(a_s) = a_{ts}$

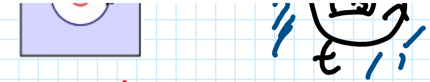
$= -2!$

- What are a_{ts}, a_{st} for V_{km}?
- Recall Cartan matrix for $\tilde{A}_1 = \begin{pmatrix} 2 & \overset{a_{ts}}{-2} \\ \underset{a_{st}}{-2} & 2 \end{pmatrix}$

$\Rightarrow$ Have 2-functor $\Sigma =$ "add white space"

$$\mathrm{Fund}^{\Omega}_{(a)}(\mathfrak{sl}_2) \xrightarrow{\Sigma} m\mathbb{SBSBim}(W_a^\vee(\mathfrak{sl}_2), V_{km})$$

<u>Thm:</u> Σ is an equivalence in deg 0



$= -2!$

 $=$  $= \partial_s(a_t) = a_{st}$

$= -2!$

Big Picture of the role of SSBim in Representation Theory 3

Thrm: In the equivalence $\mathrm{Fund}_{(q)}(\mathfrak{sl}_2) \simeq TL(-[2]_q)$ the image of $JW_n \in TL_n(-[2]_q)$ in $\mathrm{Fund}_q(\mathfrak{sl}_2)$ is the projector to the irreducible rep $L(n)$.

- Now, b/c Σ is an equivalence and since

$$[\text{simples}] \longleftrightarrow \text{KL basis}$$

this will categorify to

$$L(n) \longleftrightarrow B_{(s,t,\dots)} \leftarrow n(\pm 1?) \text{ times}$$

$\Rightarrow$ JW_n in $SSB(\emptyset,\emptyset)$ will be projector

to $B_{(s,t,\dots)}$