MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 1 SOLUTIONS

(1) For each of the following functions, state its domain, range and inverse. If the inverse does not exist, state why.

(a) $f(x) = \dfrac{2}{1 - x^3}$

Solution. This is a rational function of x, so the only way it can be undefined is if the denominator is zero. That happens only when

$$1 - x^3 = 0 \quad \Longrightarrow \quad x^3 = 1 \quad \Longrightarrow \quad x = 1.$$

So the domain is $\boxed{\text{all } x \text{ such that } x \neq 1}$. Alternatively, you can say the domain is $\boxed{(-\infty, 1) \cup (1, \infty)}$, or $\boxed{\{x \in \mathbb{R} \mid x \neq 1\}}$.

To find the range, you can either find the domain of the inverse (if the inverse exists), or think about how the range changes step-by-step as we apply each operation in the function. We'll do the former technique for this problem, and do the latter for (b), because here the inverse does exist.

To find the inverse, just follow the steps we covered in class.

(i) Write $y = 2/(1 - x^3)$.

(ii) Solve for x in terms of y:

$$y = \frac{2}{1 - x^3} \quad \Longrightarrow \quad x = \left(1 - \frac{2}{y}\right)^{1/3}.$$

(iii) Swap x and y so that the inverse is a function of x.

So the inverse is $\boxed{f^{-1}(x) = (1 - 2/x)^{1/3}}$. This is a well-defined function. Its domain is $\boxed{\{x \in \mathbb{R} \mid x \neq 0\}}$, which is also the range of the original $f(x)$.

(b) $f(x) = \ln(x^2 - 2)$

Solution. The natural logarithm ln is undefined unless its input is positive. So the domain of $f(x)$ is all x such that $x^2 - 2 > 0$. This is $\boxed{(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)}$, or $\boxed{\{x \in \mathbb{R} \mid |x| > \sqrt{2}\}}$.

To find the range, we can think about how the range changes step-by-step as you apply each operation in the function.

function	x	x^2	$x^2 - 2$	$\ln(x^2 - 2)$
range	$(-\infty, \infty)$	$[0, \infty)$	$[-2, \infty)$	$(-\infty, \infty)$.

So the range of $f(x)$ is $\boxed{(-\infty, \infty)}$, or the set $\boxed{\mathbb{R}}$ of all real numbers.

The inverse $\boxed{\text{does not exist}}$. You can see this either by plotting the function and seeing visually that the horizontal line test fails, or by explicitly giving two x values which produce the same output. For example:

$$f(3) = \ln(3^2 - 2) = \ln(6) = \ln((-3)^2 - 2) = f(-3).$$

(This method is best when you can't go and plot the function on WolframAlpha.)

(c) $f(x) = 4 + \arctan(x)$

This problem requires you to know the domain and range of $\arctan(x)$, and then to shift the range by 4. So the domain is everything, i.e. $\boxed{(-\infty, \infty)}$ or $\boxed{\mathbb{R}}$, and the range is $\boxed{(4 - \pi/2, 4 + \pi/2)}$.

The inverse exists, and you do the usual thing to find it:

$$y = 4 + \arctan(x) \qquad \Longrightarrow \qquad x = \tan(y - 4),$$

so that the inverse is $\boxed{f^{-1}(x) = \tan(x - 4)}$.

(2) The step function $H(x)$ is defined by

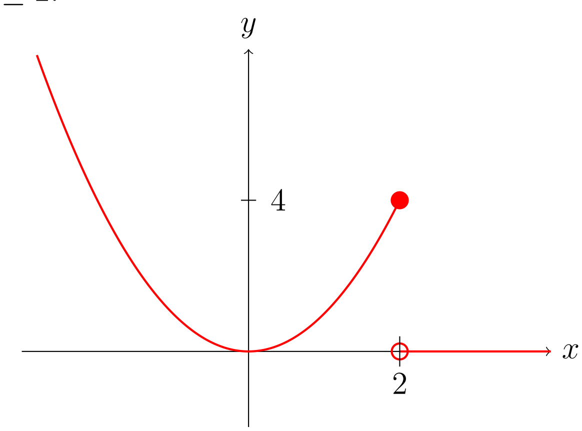
$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

(a) Sketch the function $x^2 H(4 - 2^x)$.

Solution. It is important to take this piece-by-piece. First, we must understand what $H(4 - 2^x)$ looks like. Whenever $4 - 2^x < 0$, the output of this function will be zero. (This is because the definition of H says "if the input is negative, produce the output 0".) Otherwise, when $4 - 2^x \geq 0$, the output is 1. Hence we see that

$$x^2 H(4 - 2^x) = \begin{cases} x^2 \cdot 1 & \text{if } 4 - 2^x \geq 0, \text{ i.e. } x \leq 2 \\ x^2 \cdot 0 & \text{if } 4 - 2^x < 0, \text{ i.e. } x > 2. \end{cases}$$

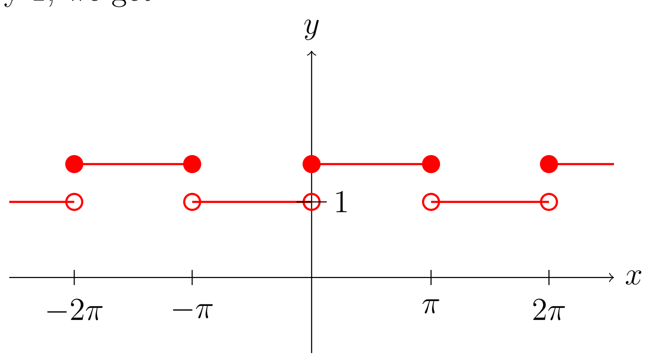
This is now easy to graph: just draw the zero function for $x > 2$, and draw the function x^2 for $x \leq 2$:



(It doesn't matter to me whether the scale is correct, but it *does* matter that you get whether the endpoints are open or closed correct!)

(b) Sketch the function $1 + H(\sin x)/2$.

Solution. Repeat the same procedure as in part (a), but with $H(\sin x)$. This function gives 1 when $\sin x$ is non-negative and 0 otherwise. So after scaling by 2 and shifting by 1, we get



(3) Write $2 + \sin(2 + \ln(\ln x))$ as some composition of *only* the following functions:

$$f(x) = 2 + x, \quad g(x) = \sin(x), \quad h(x) = \ln(x).$$

Solution. It is best to start from the inside and work your way out. For example, $\ln(\ln x)$ becomes $\ln(h(x))$, which becomes $h(h(x))$. The final answer is

$$\boxed{f(g(f(h(h(x)))))}, \quad \text{or} \quad \boxed{(f \circ g \circ f \circ h \circ h)(x)}.$$

(4) Compute the exact value for each of the following expressions. Show and explain all your steps briefly.

(a) $e^{2\ln 3}$

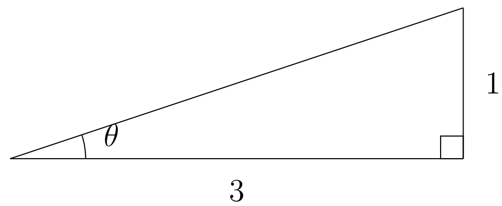
Solution. Whenever you see expressions like a^{xy}, you should think $(a^x)^y$ in an attempt to simplify. Here you should recognize that the exponential "cancels" the ln, and do

$$\boxed{e^{2\ln 3} = (e^{\ln 3})^2 = (3)^2 = 9}.$$

(I'm happy as long as you write something equivalent to this.)

(b) $\sin(\tan^{-1}(1/3))$

Solution. When in doubt, draw the appropriate triangle.



This triangle expresses that $\tan\theta = 1/3$, so that whatever $\tan^{-1}(1/3)$ is, it is equal to θ. This simplifies the problem into asking what $\sin\theta$ is. The hypotenuse has length $\sqrt{10}$, so the final answer is $\boxed{1/\sqrt{10}}$.

(5) Annoyed by your calculus homework, you crumple it into a ball and throw it into an infinitely deep hole. You observe that its speed in meters per second is given by the function

$$v(t) = 3 - \frac{1}{t+1}$$

where t is the time in seconds since you threw it. (Note: this is *not* a physically realistic model, which would be more complicated.)

(a) How fast is your homework initially traveling, right when you threw it?

Solution. This is the value of $v(t)$ at $t = 0$, i.e. $\boxed{v(0) = 2}$.

(b) What should the domain of $v(t)$ be? Explain why. (Hint: many values of t do not make sense as inputs.) What is the range of $v(t)$?

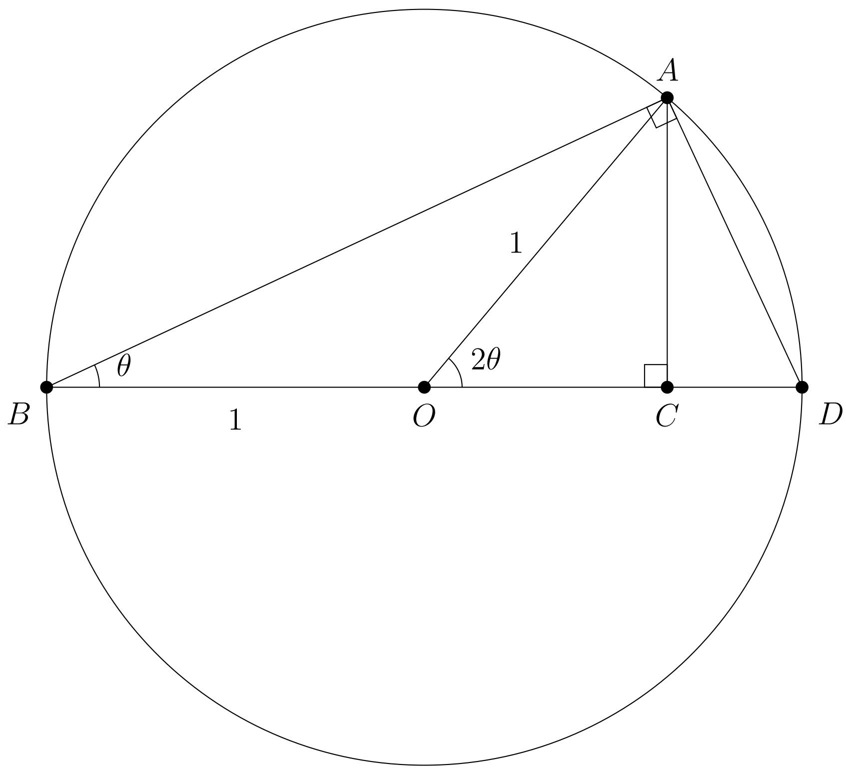
Solution. It does not make sense to plug in a negative value for time. So the domain is $\boxed{[0, \infty)}$, or $\boxed{\{x \in \mathbb{R} \mid x \geq 0\}}$. To figure out the range you can do either of the methods described in problem (1a); the final answer is $\boxed{[2, 3)}$ or $\boxed{\{x \in \mathbb{R} \mid 2 \leq x < 3\}}$.

(c) At what time t does the speed of your homework reach v meters per second?

Solution. This asks for the inverse function, i.e. for t as a function of v instead of v as a function of t. Here t and v have meanings as variables, so you just solve for t in terms of v without swapping them:

$$v = 3 - \frac{1}{t+1} \quad \Longrightarrow \quad \boxed{t = \frac{1}{3-v} - 1}.$$

(6) Use the following steps and diagram in the unit circle to compute a formula for $\sin(2\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$. Briefly explain each of your answers. A correct answer with no explanation is worth zero points.



(The circle is a unit circle, with origin O.)

(a) Using triangle OCA, what is the length of AC in terms of 2θ?

Solution. By the definition of sin, the length of AC is $\boxed{\sin(2\theta)}$.

(b) Using triangle BAD, what is the length of AB in terms of θ?

Solution. Since BD has length 2, by the definition of cos, the length of AB is $\boxed{2\cos(\theta)}$.

(c) Using triangle BCA, write $\sin\theta$ in terms of the answers of the previous two questions. Rearrange it to get the *double angle formula* for $\sin(2\theta)$. What is the formula?

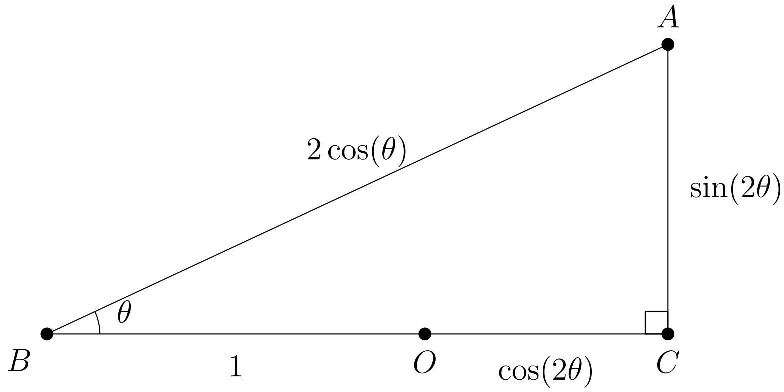
Solution. By the definition of sin and the answers to (a) and (b), we get

$$\sin\theta = \frac{AC}{AB} = \frac{\sin(2\theta)}{2\cos(\theta)}$$

Rearranging, $\boxed{\sin(2\theta) = 2\sin(\theta)\cos(\theta)}$.

(d) What is the length of OC in terms of 2θ? Use it and triangle BCA to compute $\cos(\pi/12)$ in terms of $\sin(\pi/6)$ and $\cos(\pi/6)$.

Solution. By the definition of cos, the length of OC is $\cos(2\theta)$. It makes sense to set $\theta = \pi/12$ so that $2\theta = \pi/6$. So we are assuming we know $\sin(2\theta)$ and $\cos(2\theta)$ (which we actually do), and we want to find $\cos(\theta)$. Triangle BCA has side lengths:



Hence just apply the Pythagorean theorem to get

$$\boxed{\cos(\theta) = \frac{1}{2}\sqrt{(1+\cos(2\theta))^2 + \sin(2\theta)^2}}.$$

If you plugged in values $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$, then you find after simplifying that

$$\boxed{\cos(\pi/12) = \frac{1}{2}\sqrt{2+\sqrt{3}}}.$$

(I'm happy with either as an answer.)