MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 10 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 15 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Find the most general f. (Make sure to check your answer via differentiation.)

(a) $f'(x) = \sqrt{2} + \sqrt{x}$ with $f(2) = 0$.

Solution. Using the power rule in reverse,

$$f(x) = \sqrt{2}x + \frac{2}{3}x^{3/2} + C.$$

Calculate that $f(2) = 2\sqrt{2} + (2/3)2\sqrt{2} + C = 0$. (Here $2^{3/2} = \sqrt{8} = 2\sqrt{2}$.) Solving for C gives $C = -(10/3)\sqrt{2}$. So the most general antiderivative here is

$$f(x) = \boxed{\sqrt{2}x + \frac{2}{3}x^{3/2} - \frac{10}{3}\sqrt{2}}.$$

(b) $f''(x) = 4x^3 + 1/x^2$ with $f(1) = 0$ and $f'(1) = 1$.

Solution. Again, use the power rule in reverse:

$$f'(x) = x^4 - \frac{1}{x} + C.$$

We are told $f'(1) = 1^4 - 1/1 + C = 1$, so $C = 1$. So $f'(x) = x^4 - 1/x + 1$. Now anti-differentiate again:

$$f(x) = \frac{1}{5}x^5 - \ln(x) + x + C.$$

We are told $f(1) = 1/5 - 0 + 1 + C = 0$. So $C = -6/5$. So the most general antiderivative here is

$$f(x) = \boxed{\frac{1}{5}x^5 - \ln(x) + x - \frac{6}{5}}.$$

(c)

$$f'(t) = (t+2)^9 + \frac{1}{t+2}.$$

Solution. Here we can't directly apply the power rule in reverse, but we can make a guess based on it:

$$f(t) = \boxed{\frac{1}{10}(t+2)^{10} + \ln(t+2) + C}.$$

This turns out to work (because even though we require the chain rule when differentiating, the extra factor is just 1).

(d)

$$f'(\theta) = 4\sec^2(\theta) + \frac{3}{\sqrt{1-\theta^2}}.$$

Solution. Both terms are known antiderivatives:

$$f(\theta) = \boxed{4\tan(\theta) + 3\arcsin(\theta) + C}.$$

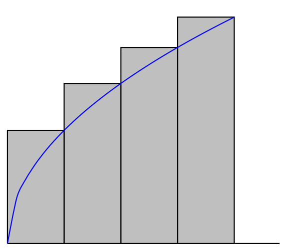
(e) $f'(x) = f(x)$ with $f(0) = 1$. (Hint: just make an educated guess and check it.)

Solution. We are looking for a function $f(x)$ whose derivative is itself. This should remind you of the exponential function e^x. Indeed, this works: $\boxed{f(x) = e^x}$ satisfies $f'(x) = e^x = f(x)$. (Check that $f(0) = e^0 = 1$.)

(2) Consider the function $f(x) = \sqrt{x}$ on the interval $[0, 1]$.

(a) Numerically approximate the area under $f(x)$ using four rectangles of equal width. Draw a diagram showing your four rectangles in relation to $f(x)$. Explain whether your approximation is smaller or larger than the actual area.

Solution. The pieces we get are $[0, 1/4], [1/4, 2/4], [2/4, 3/4], [3/4, 1]$; these are the bases of the four rectangles. You are free to use either left or right endpoints for the heights, but answers must be consistent with what you drew in your diagram. For example, using right endpoints,



$$= \frac{1}{4}\sqrt{\frac{1}{4}} + \frac{1}{4}\sqrt{\frac{2}{4}} + \frac{1}{4}\sqrt{\frac{3}{4}} + \frac{1}{4}\sqrt{\frac{4}{4}}$$

and this approximation is $\boxed{\text{larger}}$ than the actual value.

(b) Write down an expression approximating the area under $f(x)$ using n rectangles of equal width. Do *not* compute the actual value of the expression; that would be hard.

Solution. The intervals are now $[0, 1/n], [1/n, 2/n], \ldots, [(n-1)/n, 1]$. Each has width $1/n$. Using right endpoints, the height of the k-th rectangle is $\sqrt{k/n}$. So the approximation is

$$\frac{1}{n}\sqrt{\frac{1}{n}}+\frac{1}{n}\sqrt{\frac{2}{n}}+\cdots+\frac{1}{n}\sqrt{\frac{n-1}{n}}+\frac{1}{n}\sqrt{\frac{n}{n}}=\boxed{\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}}.$$

(I don't care whether or not you use summation notation.)

(c) To get an approximation which is easier to compute, use n rectangles of unequal widths. More precisely, let the base of the first rectangle be $[0, 1^2/n^2]$, the base of the second rectangle be $[1^2/n^2, 2^2/n^2]$, and so on, up to the n-th rectangle, whose base is $[(n-1)^2/n^2, 1]$.

(i) What is the width of the k-th rectangle? What about its height?

Solution. The base of the k-th rectangle is $[(k-1)^2/n^2, k^2/n^2]$, so its width is

$$\frac{k^2}{n^2}-\frac{(k-1)^2}{n^2}=\frac{k^2-(k-1)^2}{n^2}=\boxed{\frac{2k-1}{n^2}}.$$

Its height, if we use right endpoints, is

$$\sqrt{\frac{k^2}{n^2}}=\boxed{\frac{k}{n}}.$$

(ii) Write down an expression approximating the area under $f(x)$. Do compute the actual value of the expression this time. (It will depend on n.)

Solution. From (i), the area of the k-th rectangle is

$$\frac{2k-1}{n^2}\cdot\frac{k}{n}=\frac{2k^2-k}{n^3}.$$

We want to sum this up for $k=1,2,3,\ldots,n-1,n$. In other words, we want

$$\sum_{k=1}^{n}\frac{2k^2-k}{n^3}.$$

We can factor out the $1/n^3$ and rearrange the sum to get

$$\frac{1}{n^3}\left(2\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k\right)=\frac{1}{n^3}\left(2\cdot\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right).$$

This simplifies into

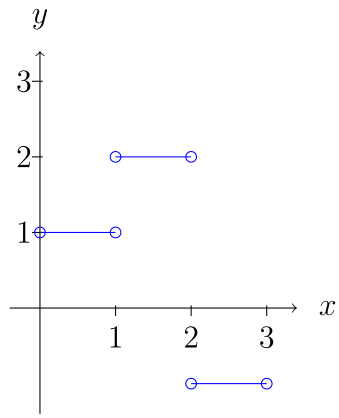
$$\boxed{\frac{n(n+1)(2n+1)}{3n^3}-\frac{n(n+1)}{2n^3}}.$$

(iii) Using (ii), what is the actual area under $f(x)$?

Solution. If we take the limit $n \to \infty$, the approximate area becomes the actual area:

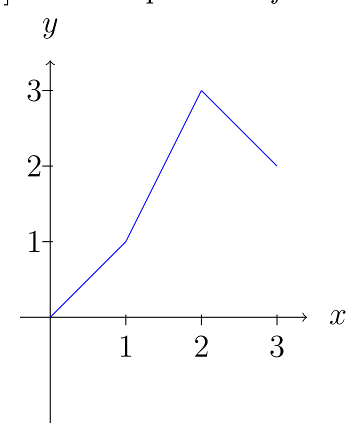
$$\lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{3n^3} - \frac{n(n+1)}{2n^3}\right) = \frac{2}{3} - 0 = \boxed{\frac{2}{3}}.$$

(3) The following is the graph of f'.



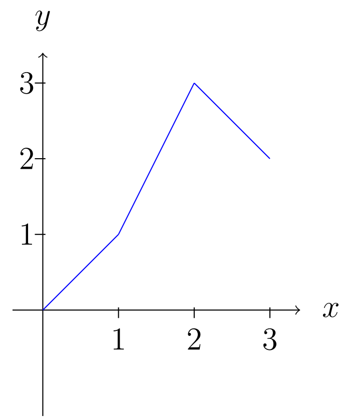
(a) Draw the graph of f, assuming $f(0) = 0$.

Solution. On $[0, 1]$, the antiderivative f is a straight line with slope 1. On $[1, 2]$ it has slope 2, and on $[2, 3]$ it has slope 3. So f looks like:



(b) Let $F(x)$ be the area under the graph of f' on the interval $[0, x]$. For example, $F(1) = 1$ and $F(2) = 1 + 2 = 3$ and $F(3) = 1 + 2 + (-1) = 2$. Draw the graph of F.

Solution. For $0 \le x \le 1$, the function f' on the interval $[0, x]$ is just a rectangle of height 1 and width x, and therefore of area x. The point is that on $[0, 1]$, the function $F(x) = x$ is just a straight line connecting $F(0) = 0$ and $F(1) = 1$. On the other intervals, F is also a straight line by the same kind of reasoning. So it looks like:



(4) Annoyed by your calculus homework, you crumple it into a ball and throw it into an infinitely deep hole.

(a) Acceleration due to gravity is $a(t) = 9.8$ m/s^2. What is the velocity of your homework as a function of t if its initial velocity when you threw it was 2 m/s? (Hint: remember that acceleration is the derivative of velocity.)

Solution. Since $a(t) = v'(t)$, its antiderivative is velocity:

$$v(t) = 9.8t + C.$$

Since the initial velocity is given as $2 = v(0) = 9.8 \cdot 0 + C$, we get $C = 2$. Hence the velocity is $v(t) = \boxed{9.8t + 2}$ m/s.

(b) How far has your homework fallen after three seconds? (Hint: remember that velocity is the derivative of distance.)

Solution. Let $x(t)$ be the distance at time t. Then $v(t) = x'(t)$. The antiderivative of $v(t)$ from (i) gives

$$x(t) = 4.9t^2 + 2t + C.$$

When $t = 0$, the total distance traveled is zero. So $x(0) = 0$, and hence $C = 0$. We get

$$x(t) = 4.9t^2 + 2t.$$

When $t = 3$, the total distance traveled is $x(3) = 4.9 \cdot 3^2 + 2 \cdot 3 = \boxed{50.1}$ m.

(c) What is the velocity of your homework after ten minutes? Does this make sense? Why or why not? (If you are more used to units of km/hr instead of m/s, convert to km/hr before judging whether your answer makes sense or not.)

Solution. Plug in $t = 600$ s to get

$$v(600) = \boxed{5882} \text{ m/s} \approx 21000 \text{ km/h}.$$

This evidently $\boxed{\text{doesn't make sense}}$: if a ball falls for 10 minutes, it does not reach anywhere close to this velocity. (The reason is that there are actually *two* contributions to acceleration: gravity, and also air resistance. In this whole problem, we have neglected air resistance.)