MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 11 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 15 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Evaluate the following definite integrals using any method.

(a)

$$\int_0^2 (2x - x^2)\,dx$$

Solution. Using the fundamental theorem of calculus,

$$\int_0^2 (2x - x^2)\,dx = \left(x^2 - \frac{x^3}{3}\right)\Big|_{x=2} - \left(x^2 - \frac{x^3}{3}\right)\Big|_{x=0} = 4 - \frac{8}{3} = \boxed{\frac{4}{3}}.$$

(b)

$$\int_{-2}^2 (1 + \sqrt{4 - x^2})\,dx$$

Solution. First split up the integral:

$$\int_{-2}^2 1\,dx + \int_{-2}^2 \sqrt{4 - x^2}\,dx.$$

The first integral is equal to 4. As for the second, it is hard to find an antiderivative for $\sqrt{4 - x^2}$. Instead, interpret the integral as the area under a semicircle of radius 2:

$$\int_{-2}^2 \sqrt{4 - x^2}\,dx = \frac{1}{2}\pi \cdot 2^2 = 2\pi.$$

So the final answer is $\boxed{4 + 2\pi}$.

(c)

$$\int_0^\pi \cos(\theta)\,d\theta$$

Solution. The antiderivative of cos is sin, so using the fundamental theorem of calculus,

$$\int_0^{\pi} \cos(\theta)\, d\theta = \sin(\pi) - \sin(0) = \boxed{0}.$$

(This makes sense because exactly half of the desired area is the negative of the other half, and they cancel.)

(d)

$$\int_3^3 \sin(x)^3 \sqrt{x^7+1}\, dx$$

Solution. It is hard to find an antiderivative for the integrand. But we don't need to, because the limits of integration leave no area under the curve. So the answer is $\boxed{0}$.

(e)

$$\int_1^6 (3f(x) - 4g(x))\, dx$$

if $\int_1^8 f(x)\, dx = 2$ and $\int_6^8 f(x)\, dx = 1$ and $\int_6^1 g(x)\, dx = 3$.

Solution. Using properties of integrals,

$$\int_1^6 (3f(x) - 4g(x))\, dx = 3\left(\int_1^8 f(x)\, dx - \int_6^8 f(x)\, dx\right) - 4\left(-\int_6^1 g(x)\, dx\right).$$

Now we just plug in the given values, to get $3(2-1) - 4(-3) = \boxed{15}$.

(f)

$$\int_0^1 (u+2)(u-1)\sqrt{u}\, du$$

Solution. Expand everything in the integrand:

$$(u+2)(u-1)\sqrt{u} = (u^2+u-2)\sqrt{u} = u^{5/2} + u^{3/2} - 2u^{1/2}.$$

Its antiderivative is

$$\frac{2}{7}u^{7/2} + \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}.$$

Using the fundamental theorem of calculus, the integral is $\boxed{2/7 + 2/5 - 4/3} = -68/105$.

(g)

$$\int_{-\pi}^{\pi} |\sin(\theta)|\, d\theta$$

Solution. The best way to do this integral is to note that $|\sin(-\theta)| = |-\sin(\theta)| = |\sin(\theta)|$, so the area on $[-\pi, 0]$ is the same as the area on $[0, \pi]$. So the whole integral is just

$$\int_{-\pi}^{\pi} |\sin(\theta)|\, d\theta = 2\int_0^{\pi} |\sin(\theta)|\, d\theta = 2\int_0^{\pi} \sin(\theta)\, d\theta.$$

(Alternatively, you can manually compute the piece on $[-\pi, 0]$. Because of the absolute value, it is necessary to split the integral into two pieces.) The remaining integral can be computed using the fundamental theorem of calculus:

$$\int_0^\pi \sin(\theta)\, d\theta = -\cos(\pi) - (-\cos(0)) = -(-1) - (-(-1)) = 2.$$

So the final answer is $\boxed{4}$.

(2) Express the limit as a definite integral, and then evaluate it.

(a)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (k/n)^2}$$

Solution. The guess is that we split up an interval into n pieces of width $1/n$ each, because of the overall factor of $1/n$ and the sum from $k = 1$ to n. Since we see a term k/n, this suggests our pieces are $[0, 1/n], [1/n, 2/n], \ldots, [(n-1)/n, 1]$. Hence we have the Riemann sum for

$$\int_0^1 \frac{1}{1+x^2}\, dx.$$

(Check this by writing down its right Riemann sum, if you are not convinced.) Now recognize the integrand as the derivative of arctan. So by the fundamental theorem of calculus,

$$\int_0^1 \frac{1}{1+x^2}\, dx = \arctan(1) - \arctan(0) = \boxed{\frac{\pi}{4}}.$$

(b)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{1+k/n}$$

Solution. We are still splitting an interval into n pieces of width $1/n$. But now we have two different choices for exactly what the pieces are:

(i) $[0, 1/n], [1/n, 2/n], \ldots, [(n-1)/n, 1]$, with integrand e^{1+x};

(ii) $[1, 1 + 1/n], [1 + 1/n, 1 + 2/n], \ldots, [1 + (n-1)/n, 2]$, with integrand e^x.

So the corresponding integral is either

$$\int_0^1 e^{1+x}\, dx \quad \text{or} \quad \int_1^2 e^x\, dx.$$

Both evaluate to $\boxed{e^2 - e}$.

(3) Let $f(x) = \sqrt{1 + x^4}$.

(a) Show that $1 \le f(x) \le 1 + x^4$ for $x \ge 0$.

Solution. If $x \ge 0$ then $1 + x^4 \ge 1$. But for $z \ge 1$ we know $\sqrt{z} \le z$. So it follows that

$$\sqrt{1 + x^4} \le 1 + x^4.$$

For the lower bound, take the square root of $1 \le 1 + x^4$.

(b) Show that $1 \le \int_0^1 f(x)\,dx \le 1.2$. (Hint: use (a).)

Solution. By properties of integrals, using (a),

$$1 = \int_0^1 1\,dx \le \int_0^1 \sqrt{1+x^4}\,dx \le \int_0^1 (1+x^4)\,dx.$$

The last integral evaluates to

$$\int_0^1 (1+x^4)\,dx = (1-0) + \left(\frac{1^5}{5} - \frac{0^5}{5}\right) = 1 + \frac{1}{5} = 1.2.$$

So we get the desired bounds.

(4) Find the derivative $f'(x)$.

(a)

$$f(x) = \int_1^x \sin^3(\theta)\cos^4(\theta)\,d\theta$$

Solution. Using the fundamental theorem of calculus,

$$f'(x) = \frac{d}{dx}\int_1^x \sin^3(\theta)\cos^4(\theta)\,d\theta = \boxed{\sin^3(x)\cos^4(x)}.$$

(b)

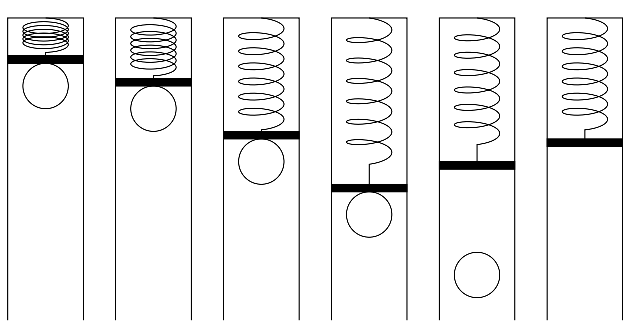
$$f(x) = \int_0^{x^2+3} (u-1)^{u-1}\,du.$$

Solution. Use the fundamental theorem of calculus, but keep the chain rule in mind:

$$f'(x) = (x^2+2)^{x^2+2} \cdot \left(\frac{d}{dx}(x^2+3)\right) = \boxed{2x(x^2+2)^{x^2+2}.}$$

(If it helps conceptually, write $f(x) = A(x^2+3)$, so that $f'(x) = A'(x^2+3)\cdot 2x$.)

(5) Annoyed by your calculus homework, you crumple it into a ball and launch it into an infinitely deep hole using the Spring Launcher Technology™ from Homework 5.



Your new and improved measurements show that at time t (in milliseconds), the end of the spring is at depth (in centimeters)

$$x(t) = -5 - \int_0^t \frac{10 \sin x}{x} \, dx.$$

(The integral is a special function called the sine integral. It is important in electrical engineering.)

(a) There are infinitely many times t where the spring will be fully extended (and about to retract back). Find all such t.

Solution. Such times t are local minimums of $x(t)$. So we must first find critical points $x'(t) = 0$. Compute

$$x'(t) = -\frac{d}{dt} \int_0^t \frac{10 \sin x}{x} \, dx = -\frac{10 \sin t}{t}.$$

This is zero exactly when the numerator is zero, i.e. $\sin t = 0$. So t can be any positive multiple of π, i.e. $n\pi$ for any positive integer n. (We never consider negative t.) To check which are local maxs vs local mins, use the second derivative test. Compute

$$x''(t) = -10 \cdot \frac{\cos(t)t - \sin(t)}{t^2}.$$

Note that $\sin(n\pi) = 0$ for any integer n, but

(i) if n is odd, then $\cos(n\pi) = -1$;

(ii) if n is even, then $\cos(n\pi) = 1$.

So $x''(n\pi)$ is positive only when n is odd. Hence the local minimums are where $\boxed{t \text{ is an odd positive multiple of } 2\pi}$.

(b) When is the first time t that the end of the spring changes from accelerating downward (i.e. extending) to accelerating upward (i.e. retracting)? You do not need to find an exact value for t; just give an equation that t must satisfy. For example: "t is the only solution to $e^{-t} = \sin(t)$ in the interval $(3, 4)$". (Hint: look back at Homework 5.)

Solution. Such times t are inflection points, i.e. $x''(t) = 0$. From the formula for $x''(t)$ above, this means we want to solve

$$\boxed{\cos(t)t - \sin(t) = 0}.$$

The first inflection point must be in between the first and second critical points, at $t = \pi$ and $t = 2\pi$. This is because the spring starts off retracting, but at some point must start expanding again to "turn around". You can check this mathematically using the intermediate value theorem:

$$\cos(\pi)\pi - \sin(\pi) = -\pi, \quad \cos(2\pi)(2\pi) - \sin(2\pi) = 2\pi,$$

so somewhere in between we must have t such that $\cos(t)t - \sin(t) = 0$. Hence $\boxed{t \text{ is the only solution to } \cos(t)t - \sin(t) = 0 \text{ in } (\pi, 2\pi)}$.