MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 12 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 15 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Compute the following indefinite integrals.

(a)

$$\int \left(\sqrt[3]{x} + \frac{1}{\sqrt{x}} \right) dx$$

Solution. Rewrite the integrand as $x^{1/3} + x^{-1/2}$. This has antiderivative

$$\boxed{\frac{3}{4} x^{4/3} + 2x^{1/2} + C}.$$

(b)

$$\int \tan^2 \theta \, d\theta$$

(Hint: $\tan = \sin / \cos$, and then use a trig identity for $\sin^2$.)

Solution. Rewrite the integrand as

$$\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1 - \cos^2 \theta}{\cos^2 \theta} = \sec^2 \theta - 1.$$

This has antiderivative

$$\boxed{\tan \theta - \theta + C}.$$

(c)

$$\int \frac{1}{\sqrt{1 - 4x^2}} \, dx$$

Solution. Note that $4x^2 = (2x)^2$. Use the substitution $u = 2x$ to make the integrand exactly the derivative of arcsin. We have $du = 2\,dx$, so after substitution the integral becomes

$$\frac{1}{2} \int \frac{1}{\sqrt{1 - u^2}} \, du = \frac{1}{2} \arcsin u + C = \boxed{\frac{1}{2} \arcsin(2x) + C}.$$

(d)

$$\int \sqrt{2 \tan t} \sec^2 t \, dt$$

Solution. Use the substitution $u = 2\tan t$. Then $du = 2\sec^2 t\, dt$, and the integral becomes

$$\frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\frac{2}{3}u^{3/2} + C = \boxed{\frac{1}{3}(2\tan t)^{3/2} + C}.$$

(e)

$$\int \frac{\sin(\ln x)}{x}\, dx$$

Solution. Use the substitution $u = \ln x$. Then $du = (1/x)\, dx$, and the integral becomes

$$\int \sin u\, du = -\cos u + C = \boxed{-\cos(\ln x) + C}.$$

(f)

$$\int x^5\sqrt{1 + x^3}\, dx$$

Solution. Use the substitution $u = 1 + x^3$. Then $du = 3x^2\, dx$ and the integral becomes

$$\frac{1}{3}\int x^3\sqrt{u}\, du.$$

This is problematic because there is still a x^3. We can rewrite it in terms of u, by using that $u - 1 = x^3$. So the integral becomes

$$\frac{1}{3}\int (u-1)\sqrt{u}\, du = \frac{1}{3}\int (u^{3/2} - u^{1/2})\, du = \boxed{\frac{1}{3}\left(\frac{2}{5}u^{2/5} - \frac{2}{3}u^{3/2}\right) + C}.$$

(2) Compute the following definite integrals.

(a)

$$\int_{-2}^{2} (x+3)\sqrt{4 - x^2}\, dx$$

(Hint: split it into two pieces, which are computed using different methods.)

Solution. Since $\int(f+g) = \int f + \int g$ (just like with differentiation), the integral splits as

$$\int_{-2}^{2} x\sqrt{4 - x^2}\, dx + \int_{-2}^{2} 3\sqrt{4 - x^2}\, dx.$$

The first integrand is an odd function, so the integral is 0. The second integral is 3 times the area of a semicircle of radius 2. So the final answer is $0 + (3/2)\pi \cdot 2^2 = \boxed{6\pi}$.

(b)

$$\int_0^2 x f(x^2)\, dx$$

if $\int_0^4 f(x)\, dx = 3$.

Solution. Use the substitution $u = x^2$ to convert the unknown integral into the known one. Then $du = 2x\,dx$ and the integral becomes

$$\frac{1}{2}\int_{0^2}^{2^2} f(u)\,du = \frac{1}{2}\int_0^4 f(x)\,dx = \boxed{\frac{3}{2}}.$$

(Remember that for definite integrals, substitution also changes the bounds of integration.)

(c)

$$\int_{-5}^{5} \frac{\sin(x)\cos^2(x)}{1+x^4}\,dx$$

Solution. It is hopeless to try to find an antiderivative for the integrand. But the integrand is an odd function: when we plug in $-x$,

(i) $1 + x^4$ is unchanged;
(ii) $\cos(x)$ is unchanged;
(iii) $\sin(-x) = -\sin(x)$ picks up an overall minus sign.

So by symmetry the whole integral is $\boxed{0}$.

(d)

$$\int_0^1 xe^{-x^2}\,dx$$

Solution. Use the substitution $u = -x^2$. Then $du = -2x\,dx$ and the integral becomes

$$-\frac{1}{2}\int_{-0^2}^{-1^2} e^u\,du = \frac{1}{2}\int_{-1}^{0} e^u\,du = \boxed{\frac{1}{2}(1-e^{-1})}.$$

(3) Water is flowing into a container at a rate of $f(t)$ liters per minute at time t. Explain in words what $\int_2^5 f(t)\,dt$ represents.

Solution. The function $f(t)$ is the rate of change of the total amount of water in the container. So the antiderivative $F(t)$ is (up to a constant) the total amount of water in the container at time t. So by FToC, $\int_2^5 f(t)\,dt = F(5) - F(2)$ represents the $\boxed{\text{total amount of water that has flowed into the container between } t = 2 \text{ and } t = 5}$.

(4) A bacteria colony starts with 200 bacteria and grows at a rate of 2^t bacteria per hour after t hours. What is the population after 1 hour?

Solution. By the same reasoning as in the previous question, the total population growth from $t = 0$ to $t = 1$ hours is given by

$$\int_0^1 2^t = \left.\frac{2^t}{\ln(2)}\right|_{t=1} - \left.\frac{2^t}{\ln(2)}\right|_{t=0} = \frac{2}{\ln(2)} - \frac{1}{\ln(2)} = \frac{1}{\ln(2)}.$$

Since we started with 200 bacteria, the final population at $t = 1$ is $\boxed{200 + 1/\ln(2)}$.

(5) Let $f(x)$ be a continuous function on $[0, 1]$.

(a) (Warmup) Use a substitution to show that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx.$$

Solution. We can turn the right hand side into the left hand side using the substitution $u = 1 - x$. Then $du = -dx$, and

$$\int_0^1 f(1-x)\,dx = -\int_{-0}^{-1} f(u)\,du = \int_0^1 f(u)\,du,$$

which (up to renaming u to x) is exactly the left hand side.

(b) Use substitution to show that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx.$$

(Hint: the obvious substitution *will* work out in the end. Use the trig identity $\sin^2 x + \cos^2 x = 1$ and don't give up.)

Solution. Let's do the left hand side integral first. Make the substitution $u = \cos x$. Then

$$du = -\sin x\,dx = -\sqrt{1-u^2}\,dx.$$

So the integral becomes

$$-\int_1^0 f(u)\frac{1}{\sqrt{1-u^2}}\,du = \int_0^1 f(u)\frac{1}{\sqrt{1-u^2}}\,du.$$

Now let's do the right hand side integral. Make the substitution $u = \sin x$. Then

$$du = \cos x\,dx = \sqrt{1-u^2}\,dx.$$

So the integral becomes

$$\int_0^1 f(u)\frac{1}{\sqrt{1-u^2}}\,du,$$

which is exactly what we got earlier for the left hand side. Hence the two sides are equal.

(c) Using (b), compute the two integrals

$$\int_0^{\pi/2} \cos^2 x\,dx, \quad \int_0^{\pi/2} \sin^2 x\,dx.$$

(Hint: their sum is really easy to compute.)

Solution. Call the two integrals A and B. If we let $f(u) = u^2$, then part (b) shows that $A = B$. But note that

$$A + B = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\,dx = \int_0^{\pi/2} 1\,dx = \pi/2.$$

So $A = B = \boxed{\pi/4}$.