MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 13 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 10 parts, for a total of 20 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Find the area enclosed by the two curves. Roughly sketch the area.

(a) $y = x^3$ and $y = x$.

Solution. First find where these two curves intersect:

$$x^3 = x \implies x(x^2 - 1) = 0 \implies x = -1, 0, 1.$$

(i) Between -1 and 0, the total area is

$$\int_{-1}^{0} (x^3 - x)\, dx = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}.$$

(ii) Between 0 and 1, the total area is

$$\int_{0}^{1} (x - x^3)\, dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

(Alternatively, note that by symmetry the two areas must be the same.) So the total area is $1/4 + 1/4 = \boxed{1/2}$.

(b) $y = \cos x$ and $y = \sin x$ on $[0, \pi]$.

Solution. These two curves intersect at $x = \pi/4$ and nowhere else. So there are two pieces.

(i) Between 0 and $\pi/4$, the total area is

$$\int_{0}^{\pi/4} (\cos x - \sin x)\, dx = (\sin(\pi/4) + \cos(\pi/4)) - (\sin(0) + \cos(0)),$$

which comes out to $\sqrt{2} - 1$.

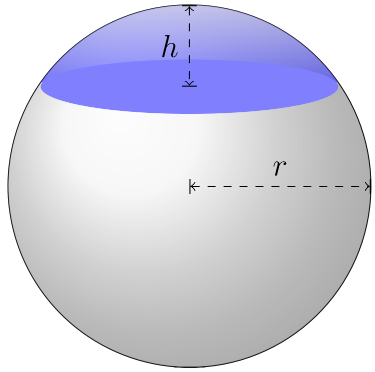
(ii) Between $\pi/4$ and π, the total area is

$$\int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx = (-\cos(\pi) - \sin(\pi)) - (-\cos(\pi/4) - \sin(\pi/4)),$$

which comes out to $\sqrt{2} + 1$.

The total area is therefore $\boxed{2\sqrt{2}}$.

(2) Consider the cap of height h in a sphere with radius r.



(a) Write an integral which computes the volume of the cap. (Hint: rotate the situation 90° first.)

Solution. The cap of height h is the solid of revolution obtained by rotating the semicircle $y = \sqrt{r^2 - x^2}$ (of radius r) on the interval $[r-h, r]$. So its volume is

$$\int_{r-h}^{r} \pi(\sqrt{r^2 - x^2})^2 \, dx = \boxed{\int_{r-h}^{r} \pi(r^2 - x^2) \, dx}.$$

(b) Explain in words what is calculated by the Riemann sum corresponding to the integral, and why it approximates the volume.

Solution. The Riemann sum adds together the volumes of little disks approximating volumes of slices of the solid of revolution. This approximates the volume for the same reason that standard Riemann sums approximate area under curves.

(c) Evaluate the integral in (a) to get the volume of the cap.

Solution. The antiderivative of $\pi r^2 - \pi x^2$ is $\pi r^2 x - (1/3)\pi x^3$. So

$$\int_{r-h}^{r} (\pi r^2 - \pi x^2) \, dx = \left(\pi r^2 \cdot r - \frac{1}{3}\pi r^3\right) - \left(\pi r^2 \cdot r - h - \frac{1}{3}\pi(r-h)^3\right).$$

After some algebra, this simplifies into

$$\boxed{\pi h^2 r - \frac{1}{3}\pi h^3}.$$

(d) Explain what answer you expect to get in (c) when $h = r$. Check that this is indeed the case.

Solution. When $h = r$, the cap is literally half of the sphere, and therefore we expect it to have volume $(1/2) \cdot (4/3)\pi r^3$. Indeed,

$$\pi r^2 r - \frac{1}{3}\pi r^3 = \boxed{\frac{2}{3}\pi r^3},$$

as expected.

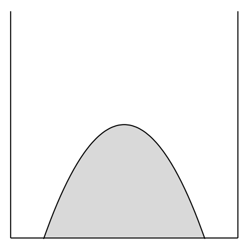
(3) Let $f(x)$ be a continuous function on $[a, b]$. By analogy with volumes of solids of revolution, make a guess for what the following integral represents:

$$\int_a^b 2\pi f(x)\, dx.$$

Explain your guess. Pick an example for $f(x)$ to illustrate why your guess is correct.

Solution. The integral represents the $\boxed{\text{surface area}}$ of the solid of revolution given by rotating f around the x axis on $[a, b]$. When we integrate $\pi f(x)^2$ to find the volume of the solid, we are essentially adding together areas of little disks. When we integrate $2\pi f(x)$, we are adding together circumferences, i.e. just the outline of little disks. These smoosh together to form the outline of the solid of revolution, i.e. the surface area.

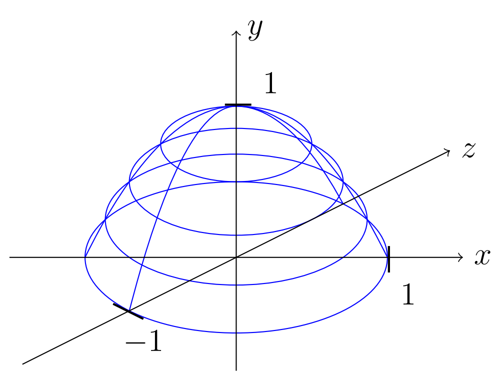
(4) After a whole semester of throwing your homework into a hole, you discover the hole is not actually infinitely deep and has a bottom! All the homeworks you threw in have formed a nice little pile at the bottom.



The pile is the solid of revolution obtained by rotating $y = 1 - x^2$ on $[0, 1]$ around *the y-axis*. We want to find its volume.

(a) Sketch a 3d diagram of the solid, with x, y, z axes labeled.

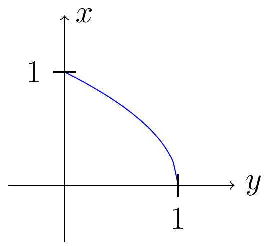
Solution. The solid is what's usually called a paraboloid:



(I don't care how well you draw it, as long as it's clear the width/length/height are 1.)

(b) Write x as a function of y, so that we can do the usual thing with solids of revolution but around the y-axis.

Solution. Solving, $\boxed{x = \sqrt{1-y}}$ for y in $[0,1]$. (This is essentially the inverse function!) It looks like



(c) Using (b), write the volume as an integral of the form $\int_0^1 f(y)\,dy$, for some function $f(y)$. Evaluate the integral to find the volume.

Solution. The volume of the solid of revolution is therefore

$$\int_0^1 \pi(\sqrt{1-y})^2\,dy = \int_0^1 (\pi - \pi y)\,dy.$$

The antiderivative is $\pi y - (1/2)\pi y^2$. So the volume is

$$\pi \cdot 1 - \frac{1}{2}\pi \cdot 1^2 = \boxed{\frac{\pi}{2}}.$$