MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 14 (BONUS)

Every question is worth 3 points. There are 20 questions, for a total of 60 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words. Because this is a bonus assignment, questions are harder than usual.

(1) For all positive integers n, prove the power rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

using the definition of a derivative. (Hint: use the binomial formula for $(x+h)^n$.)

(2) Explain why the statement is true, or give a counterexample if it is false.

- If f is continuous on an interval around a, then

$$\lim_{x \to a} |f(x)| = |f(a)|.$$

- If $|f|$ is continuous on an interval around a, then

$$\lim_{x \to a} f(x) = f(a).$$

(3) Compute the derivative

$$\frac{d^{45}}{dx^{45}}(x \sin x).$$

(4) Compute $\lim_{t\to\infty}(\sqrt{t^2+1} - t)$.

(5) What is the most general function f which satisfies the equation $f'(x) = f(x)$? What about $f''(x) = -f(x)$? (Hint: make educated guesses, and be careful where constants go.)

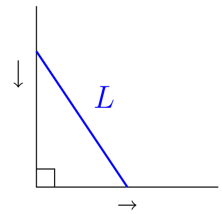
(6) The *hyperbolic* trig functions have the same relationship with the hyperbola that the usual trig functions have with the circle. Alternatively, they can be defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}.$$

Find their (first) derivatives and compare them with the derivatives of the ordinary trig functions.

(7) Differentiate $f(x) = \arccos(x) + \arcsin(x)$. What can you conclude about $f(x)$?

(8) A ladder of length L propped up against a wall is sliding downward.



If the bottom of the ladder slides away from the wall at $f(t)$ cm/s, how fast is the top of the ladder sliding downward?

(9) Explain why the model in the previous question does not make sense as the ladder finishes sliding, i.e. as the height tends to 0. How is the model flawed, and what actually happens to the ladder in real life? (Hint: try it in real life with a ruler or something analogous to a ladder.)

(10) Use the mean value theorem to prove that

$$|\sin x - \sin y| \leq |x - y|$$

for all real numbers x and y.

(11) Which points on the ellipse $5x^2 + y^2 = 3$ are farthest away from the point $(1, 0)$?

(12) Find a polynomial which has critical points at exactly 0, 1, and 2 (and nowhere else), and an inflection point at 0 (and possibly other points).

(13) Write the following integral using a Riemann sum:

$$\int_1^2 x^3\, dx.$$

Using a formula for sums of cubes, explicitly compute the Riemann sum and the value of the integral.

(14) Compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\ln(2 + k/n)}{2 + k/n}.$$

(15) Check the fundamental theorem of calculus by computing

$$\frac{d}{dx} \int_0^x \frac{t}{1 + t^4}\, dt$$

in two ways: by using the fundamental theorem of calculus, or by actually computing the integral and then differentiating.

(16) Use that $\sin(x)$ is an *increasing* function to show that $\sin(x^2) \leq \sin(x)$ whenever $0 \leq x \leq 1$. Hence find an upper bound for

$$\int_0^{\pi/6} \sin(x^2)\, dx.$$

(17) Use the substitution $u = \sin x$ to compute

$$\int \sin^4 x \cos^3 x \, dx.$$

(18) Suppose f satisfies $f(x-1) = -f(1-x)$. Compute, with explanation,

$$\int_{-1}^{3} f(x)^3 \, dx.$$

(Hint: what does the equation means in terms of symmetries of f?)

(19) Find the area of the region bounded by $y = 3|x|$ and $y = 4 - x^2$.

(20) Describe the solid of revolution whose volume is

$$\int_0^1 \pi x^2 \, dx.$$

If instead of using circular disks to approximate volume we used square disks instead, what would be the corresponding solid? What is its volume?