MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 3 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 15 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) For each of the following functions, state where it fails to be continuous and where it fails to be differentiable. At the points where it fails to be continuous or differentiable, briefly explain why it fails.

(a)

$$f(t) = \frac{2+t}{2-t}$$

Solution. There is an infinite discontinuity at $t = 2$ (so the function is also not differentiable there). It is continuous everywhere else, by the laws of continuity. To check where it is differentiable, one way is to directly compute the derivative:

$$\begin{aligned} f'(t) &= \lim_{h\to 0} \frac{\frac{2+(t+h)}{2-(t+h)} - \frac{2+t}{2-t}}{h} \\ &= \lim_{h\to 0} \frac{(2+t+h)(2-t)-(2-t-h)(2+t)}{h(2-t-h)(2-t)} \\ &= \lim_{h\to 0} \frac{4h}{h(2-t-h)(2-t)} = \lim_{h\to 0} \frac{4}{(2-t-h)(2-t)} = \frac{4}{(2-t)^2}. \end{aligned}$$

So as long as $t \neq 2$, the derivative $f'(t)$ exists. Hence the function f is $\boxed{\text{continuous and differentiable everywhere except } t = 2}$, where it fails to be continuous and therefore also fails to be differentiable.

(b)

$$f(x) = \begin{cases} -x^2 & x \leq 0 \\ 0 & 0 < x < 1 \\ x-1 & x \geq 1 \end{cases}$$

Solution. Each piece in the piecewise function is itself continuous/differentiable, so we just have to check continuity/differentiability at the endpoints.

- Check continuity at $x = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x^2 = 0 = \lim_{x\to 0^+} f(x),$$

so $\lim_{x\to 0} f(x) = 0 = f(0)$ and f is indeed continuous there.

- Check differentiability at $x = 0$:

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(-h^2)-(-0^2)}{h} = 0$$
$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{0-0}{h} = 0,$$

  so $f'(0) = 0$ exists.
- Check continuity at $x = 1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x - 1 = 0 = \lim_{x\to 1^-} f(x),$$

  so $\lim_{x\to 1} f(x) = 0 = f(1)$ and f is indeed continuous there.
- Check differentiability at $x = 1$:

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$$
$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h-1)-(1-1)}{h} = 1,$$

  so the limit defining $f'(1)$ has different left and right limits and therefore does not exist.

So f is $\boxed{\text{continuous everywhere}}$ and $\boxed{\text{differentiable everywhere except } x = 1}$.

An alternate way to check differentiability is to just compute the derivative of the pieces of f, and see if they give the same answer for $f'(0)$ and $f'(1)$. These derivatives are:

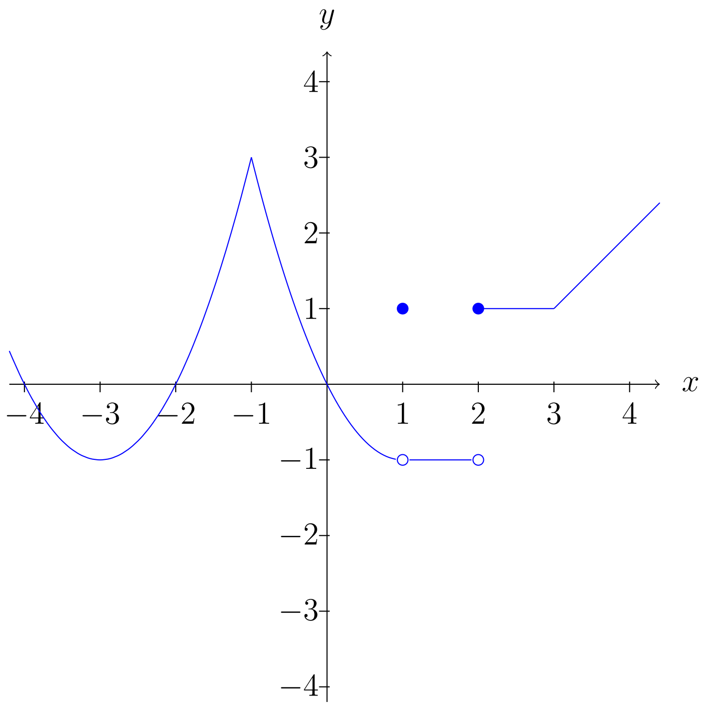
$$(-x^2)' = \lim_{h\to 0} \frac{-(x+h)^2-(-x^2)}{h} = \lim_{h\to 0} \frac{-2hx-h^2}{h} = -2x$$
$$(x-1)' = \lim_{h\to 0} \frac{(x+h-1)-(x-1)}{h} = 1.$$

Using these computations, the derivative of $f(x)$ is:

$$f'(x) = \begin{cases} -2x & x < 0 \\ ?? & x = 0 \\ 0 & 0 < x < 1 \\ ?? & x = 1 \\ 1 & x > 1. \end{cases}$$

(We don't know yet what the derivatives at $x = 0$ and $x = 1$ are; they might not even exist!) We see that $\lim_{x\to 0} f'(x)$ exists because both left and right limits are 0, but $\lim_{x\to 1} f'(x)$ does not exist because the left limit is 0 but the right limit is 1. Hence $f'(0) = 0$ and $f'(1)$ does not exist.

(c) The function $f(x)$ given by the graph:



Solution. There is a removable discontinuity at $x = 1$ and a jump discontinuity at $x = 2$. So at those points $f(x)$ also fails to be differentiable. In addition, it fails to be differentiable at $x = -1$, because the slopes from the left/right are different. The same is true at $x = 3$. So

- f is continuous $\boxed{\text{everywhere except } x = 1, 2}$;
- f is differentiable $\boxed{\text{everywhere except } x = -1, 1, 2, 3}$.

(2) (**Updated**) Compute the limit

$$\lim_{x\to 4} \ln\left(\frac{2-\sqrt{x}}{4-x}\right)$$

using that $\ln(x)$ is a continuous function. Briefly explain the step in the computation which requires this fact.

Solution. Since ln is a continuous function, we can move the limit inside, to get

$$\ln\left(\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x}\right) = \ln\left(\lim_{x\to 4} \frac{1}{2+\sqrt{x}}\right) = \boxed{\ln\frac{1}{4}}.$$

(3) You want to examine solutions to the equation $\sin(x) = \cos(x)$.

(a) Use the intermediate value theorem to show that there must exist a solution in the interval $(0, \pi/2)$.

Solution. Look at the function $f(x) = \sin(x) - \cos(x)$. Saying that $\sin(x) = \cos(x)$ has a solution at $x = a$ means that $f(a) = 0$.

- At $x = 0$, we have $\boxed{f(0) = \sin(0) - \cos(0) = -1}$.
- At $x = \pi/2$, we have $\boxed{f(\pi/2) = \sin(\pi/2) - \cos(\pi/2) = 1}$.

Since both sin and cos are continuous, $\boxed{f(x) \text{ is continuous}}$ as well. By the intermediate value theorem, $f(a) = 0$ for some $a \in (0, \pi/2)$. (The actual value is $a = \pi/4$.)

(b) Explain why this implies the equation has infinitely many solutions.

Solution. Both $\sin(x)$ and $\cos(x)$ are *periodic functions*. This means that every 2π, they repeat:

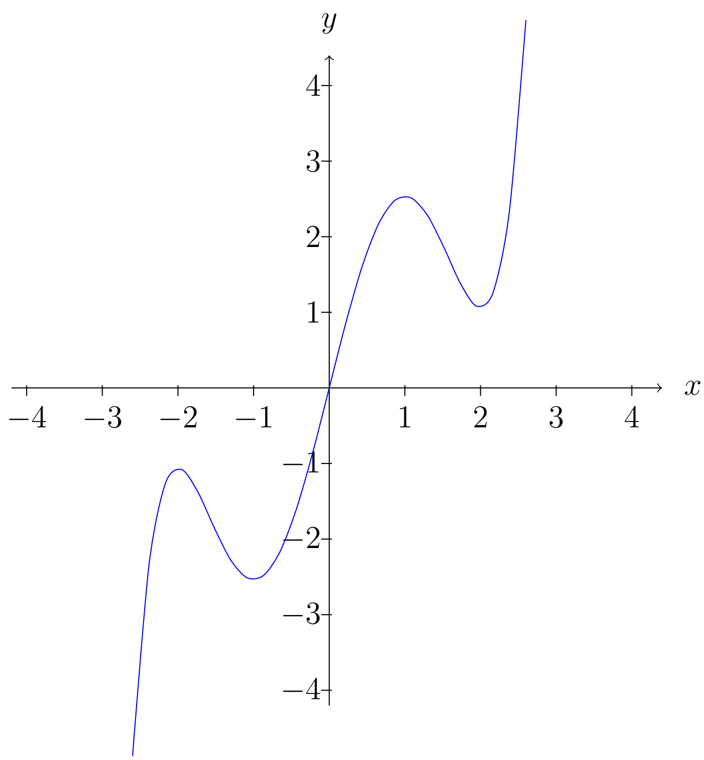
$$\sin(x) = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \cdots$$

and similarly for $\cos(x)$. So if $f(a) = 0$, then

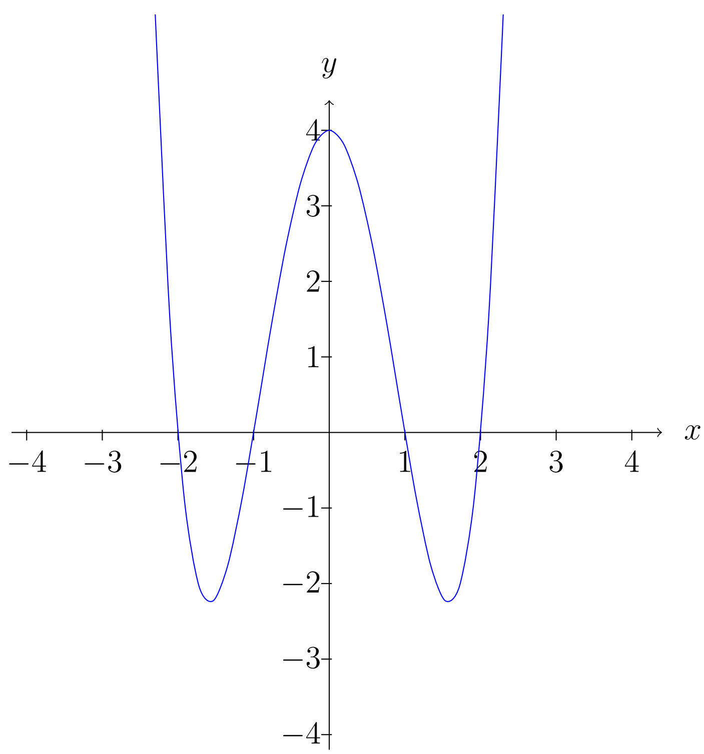
$$\boxed{0 = f(a) = f(a + 2\pi) = f(a + 4\pi) = \cdots},$$

giving infinitely many solutions.

(4) Use the following graph of the function $f(x)$ to roughly sketch the graph of its derivative $f'(x)$.



Solution. The derivative looks like:



The best way to do questions like this is to identify where $f(x)$ has slope 0, where it is increasing, and where it is decreasing.

- If $f(x)$ has slope 0, then its derivative will be 0.
- If $f(x)$ is increasing, its derivative will be positive.
- If $f(x)$ is decreasing, its derivative will be negative.

As long as your sketch doesn't violate any of the above observations, I'm happy with it.

(5) Find the equation of the tangent line to $y = 1/x + 1$ at the point $(1, 2)$.

Solution. The slope of the tangent line is given by the derivative of $f(x) = 1/x+1$, which is:

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{1}{x+h}+1\right) - \left(\frac{1}{x}+1\right)}{h} = \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x} = -\frac{1}{x^2}.$$

At $x = 1$, the slope is therefore $f'(1) = -1$. The line is therefore of the form $y = (-1)x + b$. It also passes through $(1, 2)$:

$$y = -x + b \quad \rightsquigarrow \quad 2 = -1 + b \quad \rightsquigarrow \quad b = 3.$$

Hence the tangent line is $\boxed{y = -x + 3}$.

(6) Consider the function

$$f(x) = \begin{cases} 3x - 1 & x \neq 2 \\ 6 & x = 2. \end{cases}$$

(a) Find a number δ such that if $|x - 2| < \delta$, then $|3x - 6| < \epsilon$ where $\epsilon = 0.1$.

Solution. The expression $|3x - 6| < 0.1$ means $5.9 < 3x < 6.1$. Solving for x, we get

$$1.9666\cdots \approx \frac{59}{30} < x < \frac{61}{30} = 2.0333\cdots.$$

So $-1/30 < x - 2 < 1/30$. We can take $\boxed{\delta = 1/30 \approx 0.0333\cdots}$.

(b) Find a formula for δ (in terms of ϵ) such that if $0 < |x-2| < \delta$, then $|3x-6| < \epsilon$. (This formula should give the answer you got in (a) when you plug in $\epsilon = 0.1$.)

Solution. In general, if $|3x - 6| < \epsilon$, then dividing both sides by 3 gives $|x - 2| < \epsilon/3$. So we set $\boxed{\delta = \epsilon/3}$. Then indeed

$$\text{if} \quad 0 < |x - 2| < \delta \quad \text{then} \quad |3x - 6| < 3\delta = \epsilon.$$

(c) From (b), what can you conclude about $\lim_{x\to 2} f(x)$? Briefly explain why. (Hint: the ϵ/δ definition involves the condition $|f(x) - 5| < \epsilon$, where 5 is your guess for the limit $\lim_{x\to 2} f(x)$.)

Solution. According to the ϵ/δ definition, the claim that "$\lim_{x\to 2} f(x) = 5$ means that given any $\epsilon > 0$, you must be able to find $\delta > 0$ such that

$$\text{if} \quad 0 < |x-2| < \delta \quad \text{then} \quad |f(x) - 5| < \epsilon.$$

In this case, $|f(x) - 5| = |(3x-1) - 5| = |3x - 6|$. We can substitute $3x - 1$ for $f(x)$ because $0 < |x - 2|$, i.e. $x \neq 2$. But part (b) proves this exact statement! So $\boxed{\text{part (b) proves that } \lim_{x\to 2} f(x) = 5}$.

(7) Annoyed by your calculus homework, you crumple it into a ball and throw it into an infinitely deep hole. As you watch it fall, a physicist passing by says to you "the depth $x(t)$ that your homework is at, as a function of the time t since you threw it, is given by

$$x(t) = t^{3/2},$$

now stop staring at your hole and go do the next question". You decide to show that their proposed model is wrong.

(a) Using the proposed model, compute the velocity $v(t)$ of your homework at time t. (Hint: remember that velocity is the rate of change of position over time.)

Solution. We just need to compute the derivative

$$v(t) = x'(t) = \lim_{h\to 0} \frac{(t+h)^{3/2} - t^{3/2}}{h}.$$

First recognize that we can rationalize the numerator, to get

$$\begin{aligned}
\frac{(t+h)^{3/2} - t^{3/2}}{h} \frac{(t+h)^{3/2} + t^{3/2}}{(t+h)^{3/2} + t^{3/2}} &= \frac{(t+h)^3 - t^3}{h((t+h)^{3/2} + t^{3/2})} \\
&= \frac{3ht^2 + 3h^2t + h^3}{h((t+h)^{3/2} + t^{3/2})} \\
&= \frac{3t^2 + 3ht + h^2}{(t+h)^{3/2} + t^{3/2}}.
\end{aligned}$$

As $h \to 0$, this becomes

$$x'(t) = \frac{3t^2}{t^{3/2} + t^{3/2}} = \frac{3t^2}{2t^{3/2}} = \boxed{\frac{3}{2}t^{1/2}}.$$

(b) Using the proposed model, compute the acceleration $a(t)$ of your homework at time t. (Hint: remember that acceleration is the rate of change of velocity over time.)

Solution. We need to differentiate $v(t)$:

$$a(t) = v'(t) = \frac{3}{2} \lim_{h\to 0} \frac{(t+h)^{1/2} - t^{1/2}}{h}.$$

Repeat the same process as above:

$$\frac{3}{2}\frac{(t+h)^{1/2}-t^{1/2}}{h}\frac{(t+h)^{1/2}+t^{1/2}}{(t+h)^{1/2}+t^{1/2}} = \frac{3}{2}\frac{(t+h)-t}{h((t+h)^{1/2}+t^{1/2})}$$
$$= \frac{3}{2}\frac{1}{(t+h)^{1/2}+t^{1/2}}.$$

As $h \to 0$, this becomes

$$v'(t) = \frac{3}{2}\frac{1}{t^{1/2}+t^{1/2}} = \boxed{\frac{3}{4}t^{-1/2}}.$$

(c) What should the domains of $x(t)$, $v(t)$ and $a(t)$ be?

Solution. The domain of $x(t)$ should be $\boxed{[0,\infty)}$, because it does not make sense to plug in a negative value for time. The domains of $v(t)$ and $a(t)$ are a little tricky. They cannot be defined on a bigger domain than $x(t)$ is. But in addition, we see that $a(t)$ is undefined at $t = 0$. So the domain of $v(t)$ is $\boxed{[0,\infty)}$ and the domain of $a(t)$ is $\boxed{(0,\infty)}$.

(d) As $t \to 0^+$, what happens to the acceleration of your homework? Explain why this makes no sense physically, and therefore invalidates the proposed model.

Solution. As $t \to 0^+$, we have

$$\boxed{\lim_{t\to 0^+} a(t) = \lim_{t\to 0^+} \frac{1}{2\sqrt{t}} = \infty}.$$

This doesn't make any physical sense because in the process of throwing an object, you are accelerating the object a *finite* amount. In other words, the $\boxed{\text{acceleration of the object cannot ever be infinite}}$ like the model predicts.