MATH UN1101
CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 4 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 10 parts, for a total of 20 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Differentiate the following functions. State the differentiation rule used at each step.

(a) $f(x) = x^{30} - x\sqrt{x} - 2^{10}$.

Solution. Use the sum/difference rule to split the problem into simpler pieces.

- Derivative of x^{30} is $30x^{29}$, by the power rule.
- Derivative of $x\sqrt{x} = x^{3/2}$ is $(3/2)x^{1/2}$, again by the power rule.
- Derivative of 2^{10} is 0, since it is a constant.

By the sum and difference rules, the derivative is

$$f'(x) = \boxed{30x^{29} - \frac{3}{2}x^{1/2}}.$$

(b)

$$g(t) = \frac{t+1}{t^2-2}.$$

Solution. Use the quotient rule. Let the numerator be $u(t)$ and the denominator be $v(t)$. First, to avoid confusion, compute

$$u'(t) = 1, \quad v'(t) = 2t$$

using the power rule (and sum/difference rules). Then the quotient rule says that

$$g'(t) = \frac{u'(t)v(t) - u(t)v'(t)}{v(t)^2} = \frac{1 \cdot (t^2-2) - (t+1) \cdot 2t}{(t^2-2)^2} = \boxed{\frac{-t^2-2t-2}{(t^2-2)^2}}.$$

(c)

$$h(z) = \frac{Az^5 + Bz^3 + C/z}{z^{10}}$$

where A, B, C are constants.

Solution. It is best to simplify this expression first:

$$h(z) = Az^{-5} + Bz^{-7} + Cz^{-11}.$$

Now we can directly apply the sum rule to split the problem into simpler pieces.

- Derivative of Az^{-5} is A times the derivative of z^{-5}, which by the power rule is $-5z^{-6}$. So the derivative is $-5Az^{-6}$.
- Derivative of Bz^{-7} is $-7Bz^{-8}$ by the same procedure.
- Derivative of Cz^{-11} is $-11Cz^{-12}$ by the same procedure.

By the sum rule, the derivative is

$$h'(z) = \boxed{-5Az^{-6} - 7Bz^{-8} - 11Cz^{-12}}.$$

(d)

$$F(x) = (x - 5x^3)\left(\frac{2}{x^2} + \frac{1}{x^4}\right).$$

Solution. Again, it is best to simplify this expression first:

$$F(x) = (x-5x^3)(2x^{-2}+x^{-4}) = (2x^{-1}+x^{-3})-(10x+5x^{-1}) = -10x-3x^{-1}+x^{-3}.$$

Differentiate this using the same procedure as in (c), to get

$$F'(x) = \boxed{-10 + 3x^{-2} - 3x^{-4}}.$$

(e)

$$f(x) = \frac{1 + xg(x)}{\sqrt{x}}$$

where $g(x)$ is a differentiable function.

Solution. Again, it is best to simplify this expression first:

$$f(x) = x^{-1/2} + x^{1/2}g(x).$$

Use the sum rule to split it into two pieces.

- The derivative of $x^{-1/2}$ is $(-1/2)x^{-3/2}$, by the power rule.
- Use the product rule for $x^{1/2}g(x)$. Its derivative is:

$$\frac{1}{2}x^{-1/2} \cdot g(x) + x^{1/2} \cdot g'(x).$$

Putting this together, the sum rule gives

$$f'(x) = \boxed{-\frac{1}{2}x^{-3/2} + \frac{1}{2}x^{-1/2}g(x) + x^{1/2}g'(x)}.$$

(2) We want a formula for derivatives of functions like $f(x) = (g(x))^n$ for integers $n \geq 0$.

(a) Let f_1, f_2, f_3 be differentiable functions. Use the product rule twice to show that the derivative of their product is

$$(f_1 f_2 f_3)' = f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3'.$$

Solution. Treat $f_1 f_2$ as one function, and f_3 as another. Applying the product rule this way, we get

$$(f_1 f_2 \cdot f_3)' = (f_1 f_2)' f_3 + (f_1 f_2) \cdot f_3'.$$

Use the product rule again to compute $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$. Plug this back in to get:

$$\boxed{(f_1 f_2 \cdot f_3)' = (f_1' f_2 + f_1 f_2') f_3 + (f_1 f_2) \cdot f_3'}.$$

This is exactly the given formula.

(b) Let $f_1, f_2, \ldots, f_n$ be differentiable functions. Guess a formula for

$$(f_1 f_2 \cdots f_n)',$$

and briefly explain your guess.

Solution. To get general formulas, try specific examples until you see a pattern.

- When $n = 2$, the formula is the usual product rule
$$(f_1 f_2)' = f_1' f_2 + f_1 f_2'.$$
- When $n = 3$, the formula is what we got in (a):
$$(f_1 f_2 f_3)' = f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3'.$$
- When $n = 4$, we repeat the procedure from (a), to get
$$(f_1 f_2 f_3 f_4)' = (f_1 f_2 f_3)' f_4 + (f_1 f_2 f_3) f_4'.$$
But we have a formula for $(f_1 f_2 f_3)'$. Plugging it in and simplifying, we get
$$(f_1 f_2 f_3 f_4)' = f_1' f_2 f_3 f_4 + f_1 f_2' f_3 f_4 + f_1 f_2 f_3' f_4 + f_1 f_2 f_3 f_4'.$$

I think the pattern is fairly clear from here. For general n, just write down n terms of the form $f_1 f_2 f_3 \cdots f_{n-1} f_n$, and in the first term differentiate just f_1, in the second term differentiate just f_2, and so on. In other words,

$$\boxed{\begin{aligned}(f_1 f_2 \cdots f_n)' = \quad & f_1' f_2 f_3 f_4 \cdots f_{n-1} f_n \\ & + f_1 f_2' f_3 f_4 \cdots f_{n-1} f_n \\ & + f_1 f_2 f_3' f_4 \cdots f_{n-1} f_n \\ & + \cdots \\ & + f_1 f_2 f_3 f_4 \cdots f_{n-1}' f_n \\ & + f_1 f_2 f_3 f_4 \cdots f_{n-1} f_n'.\end{aligned}}$$

(c) In the special case where $f_1(x) = f_2(x) = \cdots = f_n(x) = x$, we know from the power rule that

$$(f_1(x) f_2(x) \cdots f_n(x))' = (x^n)' = n x^{n-1}.$$

Check that your guessed formula indeed produces this answer.

Solution. The key point is that when all the f_i are equal, every one of those n terms above becomes the same. In particular, for $f_1(x) = \cdots = f_n(x) = x$, we have $f_i'(x) = 1$ by the power rule, so that

$$\boxed{f_1' f_2 f_3 f_4 \cdots f_{n-1} f_n = 1 \cdot \underbrace{x \cdot x \cdot x \cdots x \cdot x}_{n-1 \text{ times}} = x^{n-1}},$$

and there are n such terms for a total of

$$(f_1 f_2 \cdots f_n)' = \boxed{n \cdot x^{n-1}},$$

as we wanted. (The other terms just multiply the 1 in a different position, which makes no difference.)

(d) In the special case where $f_1(x) = f_2(x) = \cdots = f_n(x) = g(x)$, i.e. are all equal to the same differentiable function $g(x)$, what does your formula produce? In other words, what is the formula for the derivative of $g(x)^n$?

Solution. Analogously to (c), all of the n terms are still the same. Now they are each:

$$f_1' f_2 f_3 f_4 \cdots f_{n-1} f_n = g'(x) \cdot \underbrace{g(x) g(x) \cdots g(x) g(x)}_{n-1 \text{ times}} = g'(x) g(x)^{n-1}.$$

There are n such terms, for a total of

$$(g(x)^n)' = \boxed{n g'(x) g(x)^{n-1}}.$$

(e) Use the formula from (d) to calculate the derivative of $f(x) = (3x^2 + x - 7)^{100}$.

Solution. Here $n = 100$, with $g(x) = 3x^2 + x - 7$ and therefore $g'(x) = 6x + 1$ by sum and power rules. Using the formula from (d), we get

$$f'(x) = \boxed{100 \cdot (6x + 1) \cdot (3x^2 + x - 7)^{99}}.$$