MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 6 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 15 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Differentiate the following functions.

(a)

$$g(u) = e^{3u+7}\sin(u)$$

Solution. First use the product rule:

$$g'(u) = (e^{3u+7})'\sin(u) + e^{3u+7}(\sin u)'.$$

To differentiate e^{3u+7}, use the chain rule:

$$(e^{3u+7})' = e^{3u+7}\cdot(3u+7)' = 3e^{3u+7}.$$

To differentiate $\sin(u)$, remember we showed in class the answer is $\cos(u)$. So the final answer is

$$g'(u) = \boxed{3e^{3u+7}\sin(u) + e^{3u+7}\cos(u)}.$$

(b)

$$f(x) = \left(\frac{2^x - 1}{x^2+1}\right)^{100}$$

Solution. First use the chain rule to get

$$f'(x) = 100\left(\frac{2^x-1}{x^2+1}\right)^{99}\left(\frac{2^x-1}{x^2+1}\right)'.$$

Now focus on the remaining piece. Use the quotient rule to get

$$\left(\frac{2^x-1}{x^2+1}\right)' = \frac{(2^x-1)'(x^2+1) - (2^x-1)(x^2+1)'}{(x^2+1)^2}.$$

To differentiate $2^x - 1$, use that $(b^x)' = b^x \ln b$ to get $(2^x-1)' = 2^x\ln(2)$. So the final answer is

$$f'(x) = \boxed{100\left(\frac{2^x-1}{x^2+1}\right)^{99}\frac{2^x\ln(2)(x^2+1) - 2x(2^x-1)}{(x^2+1)^2}}.$$

(I will not ask you to simplify answers any more beyond something like this.)

(c)

$$f(\theta) = \tan^2(n\theta)$$

where n is a constant.

Solution. Use the chain rule. Note that the outside function is $f(u) = u^2$, and the inside function is $g(\theta) = \tan(n\theta)$. So

$$f'(\theta) = 2\tan(n\theta) \cdot (\tan(n\theta))'.$$

To find $(\tan(n\theta))'$, use chain rule again. Remember that $(\tan u)' = \sec^2(u)$. So

$$(\tan(n\theta))' = \sec^2(n\theta) \cdot n.$$

The final answer is

$$f'(\theta) = \boxed{2n\tan(n\theta)\sec^2(n\theta)}.$$

(d)

$$h(x) = \ln(x^2 - 1)$$

Solution. Use the chain rule, remembering that $(\ln u)' = 1/u$:

$$h'(x) = \frac{1}{x^2-1} \cdot (x^2-1)' = \boxed{\frac{2x}{x^2-1}}.$$

(e)

$$h(x) = (\cos x)^{\sqrt{x}} \cdot \sqrt{x^3+7}$$

Solution. Functions like x^x require logarithmic differentiation. So begin with taking logs of both sides:

$$\ln h(x) = \sqrt{x}\ln(\cos x) + \frac{1}{2}\ln(x^3+7).$$

Now differentiate both sides, keeping the chain rule in mind:

$$\frac{1}{h(x)} \cdot h'(x) = \frac{1}{2\sqrt{x}}\ln(\cos x) + \sqrt{x}\frac{1}{\cos x} \cdot (-\sin x) + \frac{1}{2}\frac{1}{x^3+7} \cdot 3x^2.$$

Rearrange and simplify a little to get

$$h'(x) = \boxed{h(x)\left(\frac{\ln(\cos x)}{2\sqrt{x}} - \frac{\sqrt{x}\sin(x)}{\cos(x)} + \frac{3x^2}{2(x^3+7)}\right)}.$$

(I don't care whether or not you plug in the expression for $h(x)$.)

(2) Find y'' by implicit differentiation.

(a)

$$x^2 + 2y^3 = 7$$

Solution. Find y' first:

$$2x + 6y^2 \cdot y' = 0 \quad \Longrightarrow \quad y' = \frac{-2x}{6y^2} = -\frac{1}{3}xy^{-2}.$$

Now differentiate this again using product rule, keeping the chain rule in mind:

$$y'' = -\frac{1}{3}(xy^{-2})' = -\frac{1}{3}\left(1 \cdot y^{-2} + x \cdot (-2y^{-3}) \cdot y'\right).$$

Plug the formula for y' back in to get

$$y'' = \boxed{-\frac{1}{3}\left(y^{-2} - 2xy^{-3}\left(-\frac{1}{3}xy^{-2}\right)\right) = -\frac{1}{3y^2} - \frac{2x^2}{9y^5}}.$$

(I don't mind if you didn't do this last step, but be aware that future exams may explicitly ask you to.)

(b)

$$\sin x + \cos y = 4$$

Solution. Find y' first:

$$\cos(x) - \sin(y) \cdot y' = 0 \qquad \Longrightarrow \qquad y' = \frac{-\cos(x)}{-\sin(y)} = \frac{\cos x}{\sin y}.$$

Now differentiate again using quotient rule, keeping the chain rule in mind:

$$y'' = \frac{-\sin(x) \cdot \sin(y) - \cos(x) \cdot \cos(y) \cdot y'}{\sin^2(y)}.$$

Plug the formula for y' back in to get

$$y'' = \boxed{\frac{-\sin(x)\sin(y) - \cos(x)\cos(y)\frac{\cos(x)}{\sin(y)}}{\sin^2(y)}}.$$

(3) Suppose $f(x)$ has an inverse function $f^{-1}(x)$, and both of them are differentiable.

(a) Find a formula for the derivative of $f^{-1}(x)$ by using implicit differentiation and the chain rule on the equation

$$f(f^{-1}(x)) = x$$

and then rearranging. (Hint: it may help, conceptually, to temporarily rename $f^{-1}(x)$ to something like $g(x)$.)

Solution. Differentiating both sides gives

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1.$$

We want the derivative of $f^{-1}(x)$, i.e. rearrange to solve for

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}.$$

(b) Using your formula from (a), find the derivative of $\arctan(x)$. (Note: no need to simplify anything yet.)

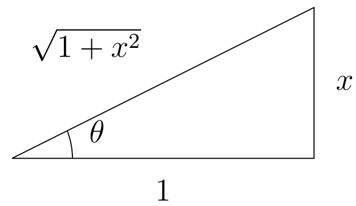
Solution. Let $f(x) = \tan(x)$ so that $f^{-1}(x) = \arctan(x)$. Recall that $f'(x) = \sec^2(x)$. Then the formula says

$$(\arctan x)' = \boxed{\frac{1}{\sec^2(\arctan(x))}}.$$

(c) Explain how to simplify your formula from (b) to get the final answer

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}.$$

Solution. Draw a right angled triangle, with unit hypotenuse, with angle $\theta = \arctan(x)$, i.e. such that $\tan(\theta) = x/1$:



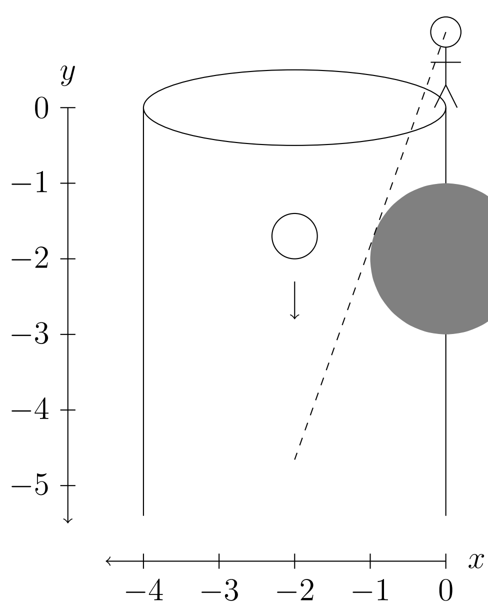
It follows that

$$\sec^2(\arctan(x)) = \sec^2(\theta) = \left(\frac{1}{\cos(\theta)}\right)^2 = \left(\frac{1}{1/\sqrt{1+x^2}}\right)^2 = 1+x^2.$$

Hence we get the formula

$$(\arctan x)' = \frac{1}{\sec^2(\arctan(x))} = \frac{1}{1+x^2}.$$

(4) Annoyed by your calculus homework, you crumple it into a ball and throw it into an infinitely deep hole. Standing at the edge of the hole, you watch your homework fall. Unfortunately, there is a rock sticking out into the hole, which prevents you from seeing past a certain depth.



Conveniently, the rock is perfectly circular and is described by the equation

$$x^2 + (y+2)^2 = 1.$$

Your homework is falling vertically along the line $x = -2$, and you would like to figure out how deep it can fall before you can no longer see it.

(a) Your deepest line of sight (the dashed line in the diagram) is tangent to the circular rock at some point (a, b). Using implicit differentiation, find the slope of the tangent line at (a, b). (Your answer will be some expression involving the variables a and b.)

Solution. Take the derivative of both sides of the equation of the circular rock:

$$2x + 2(y+2) \cdot y' = 0 \qquad \Longrightarrow \qquad y' = -\frac{x}{y+2}.$$

We want the slope at $(x, y) = (a, b)$, so plug it in to get the slope $\boxed{-a/(b+2)}$.

(b) Without using your formula from (a), and using your understanding of circles, what should be the slope at $(0, -1)$? At $(-1, -2)$? Do these answers agree with what your formula from (a) produces?

Solution. The tangent line at $(0, -1)$ is horizontal, so we expect slope 0. Indeed, the slope is $-0/(-1+2) = 0$. The tangent line at $(-1, -2)$ is vertical, so we expect slope ∞. Indeed, the slope is $1/(-2+2) \to \infty$.

(c) Your line of sight starts at $(0, 1)$, where your head is. Its slope is determined, in terms of variables a and b, by your formula from (a). Using these pieces of information, what is the equation of the line? Write your answer in the form

$$y = (\text{some expression involving } a \text{ and } b) \cdot x + (\text{some constant}).$$

Solution. We already know the slope of the tangent line, so the equation is

$$y = \frac{-a}{b+2}x + C$$

for some constant C. Since $(x, y) = (0, 1)$ lies on the line, plug it in to get

$$1 = \frac{-a}{b+2} \cdot 0 + C \qquad \Longrightarrow \qquad C = 1.$$

So the equation of the line is $\boxed{y = -ax/(b+2) + 1}$.

(d) Your line of sight must also pass through the point of tangency (a, b). Plug this fact into (c) to get an equation that a and b must satisfy. Then write down a second equation that a and b must satisfy, coming from the fact that (a, b) is a point on the circular rock. (Hint: you should end up with a system of two quadratic equations for a and b.)

Solution. Since (a, b) also lies on the tangent line, we must have

$$b = \frac{-a}{b+2} \cdot a + 1.$$

Since (a, b) lies on the circle, we also must have

$$a^2 + (b+2)^2 = 1.$$

Rearranging a little, this gives a system of two quadratic equations:

$$\boxed{\begin{cases} (b-1)(b+2)+a^2=0 \\ (b+2)^2+a^2-1=0. \end{cases}}.$$

(e) Solve for a and b, and therefore for the equation of your line of sight. How deep does your homework fall before you can't see it anymore?

Solution. From the second equation, we get $a^2 = 1-(b+2)^2$. Plug this into the first equation to get

$$(b-1)(b+2)+1-(b+2)^2=0.$$

Expanding and simplifying, we see this isn't actually a quadratic equation:

$$(b^2+b-2)+1-(b^2+4b+4)=b-1-4b-4=-3b-5.$$

Hence $b=-5/3$. Then

$$a=\sqrt{1-(b+2)^2}=\sqrt{1-(1/3)^2}=\frac{2\sqrt{2}}{3}.$$

The equation of your line of sight is therefore

$$y=\frac{-2\sqrt{2}/3}{-5/3+2}x+1=\boxed{-2\sqrt{2}x+1}.$$

Since your homework is falling along the line $x=-2$, you want to find where this line intersects $x=-2$. Plugging in $x=-2$, we get

$$y=-2\sqrt{2}\cdot 2+1=-4\sqrt{2}+1.$$

So you will not be able to see your homework beyond the depth $\boxed{-4\sqrt{2}+1}$.