MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 7 SOLUTIONS

Each part (labeled by letters) of every question is worth 3 points. There are 10 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Two quantities f and g both depend on time t and are related by the equation

$$f^4 + \sin(g) = (t+1)^2.$$

Find an equation relating f' and g'. If $f(0) = 1$ and $g(0) = 0$ and $g'(0) = 6$, what is $f'(0)$?

Solution. Implicitly differentiate the equation with respect to t to get

$$4f^3 \cdot f' + \cos(g) \cdot g' = 2(t+1).$$

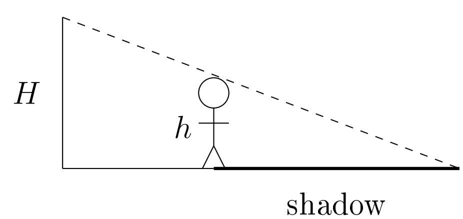
To find $f'(0)$, plug in $t = 0$:

$$4f(0)^3 \cdot f'(0) + \cos(g(0)) \cdot g'(0) = 2(0+1).$$

Using the given values, we solve and get

$$4f'(0) + \cos(0) \cdot 6 = 2 \qquad \implies \boxed{f'(0) = -1}.$$

(2) You are walking away from a streetlight and wondering how fast your shadow grows.



Let H be the height of the streetlight, and h be your own height. Write down an equation relating how far you are from the streetlight and how long your shadow is. If you are walking at 1 m/s, how fast is your shadow growing?

Solution. At time t, let $x(t)$ be how far you are from the streetlight, and $s(t)$ be the length of your shadow. Since the two triangles are similar,

$$\frac{h}{s(t)} = \frac{H}{x(t) + s(t)}.$$

(Alternatively, they are both equal to $\tan(\theta)$.) Rearranging,

$$hx(t) + hs(t) = Hs(t).$$

Differentiate this with respect to t:

$$hx'(t) + hs'(t) = Hs'(t).$$

We know $x'(t) = 1$, since you are walking at 1 m/s. So plug this in and solve for $s'(t)$ to get

$$\boxed{s'(t) = \frac{h}{H - h}}.$$

(3) Tired of calculus homework, you go ride a ferris wheel. A sign proclaims "biggest ferris wheel ever: height 120 meters". One minute after you get on the ride, you have rotated a quarter of the way, and you are already bored. For fun, you decide to compute the speed at which you are rising when you reach 90 meters above ground level. What is the speed? (Assume the ferris wheel rotates at a constant rate.)

Solution. To compute the rate of change in height h, we relate h and the angle of rotation θ. Suppose that when you get on the ferris wheel, $\theta = 0$. Then

$$h = -60\cos(\theta) + 60.$$

Differentiating both sides with respect to time,

$$\frac{dh}{dt} = 60\sin(\theta) \cdot \frac{d\theta}{dt}.$$

The ferris wheel rotates at a constant rate of one full revolution per four minutes, i.e.

$$\frac{d\theta}{dt} = 2\pi/(4 \cdot 60) \text{ rad/s}.$$

When you are 90 meters high, $\theta = 2\pi/3$. Hence

$$\frac{dh}{dt} = 60\sin\left(\frac{2\pi}{3}\right) \cdot \frac{\pi}{2 \cdot 60} = \boxed{\frac{\pi}{2}\frac{\sqrt{3}}{2}}.$$

(4) Use linear approximation to estimate $\sqrt[4]{10001}$ and $\sin(29°)$.

Solution. Let $f(x) = \sqrt[4]{x} = x^{1/4}$. Then $f'(x) = (1/4)x^{-3/4}$, and

$$\sqrt[4]{10000 + 1} \approx \sqrt[4]{10000} + \frac{1}{4 \cdot 10000^{3/4}} \cdot 1 = \boxed{10 + \frac{1}{4000}}.$$

Let $g(x) = \sin(x)$. Remember we always have to measure angles in radians. Then $g'(x) = \cos(x)$ and

$$\sin\left(\frac{\pi}{6} - \frac{\pi}{180}\right) \approx \sin\frac{\pi}{6} - \cos\left(\frac{\pi}{6}\right) \cdot \frac{\pi}{180} = \boxed{\frac{1}{2} - \frac{\sqrt{3}}{2}\frac{\pi}{180}}.$$

(5) Consider the function

$$f(x) = \ln(x^2 + x + 1)$$

on the interval $[-1, 1]$. Explain why an absolute maximum must exist on this interval. Find both the absolute maximum and absolute minimum values.

Solution. The interval $[-1, 1]$ is closed, so the function f must attain an absolute maximum by the $\boxed{\text{extreme value theorem}}$. To find where the absolute maximum and absolute minimum are, compute

$$f'(x) = \frac{1}{x^2 + x + 1} \cdot (2x + 1).$$

If $f'(x) = 0$, then the numerator is zero, i.e. $2x + 1 = 0$. Hence $x = -1/2$ is a critical number. Calculate that

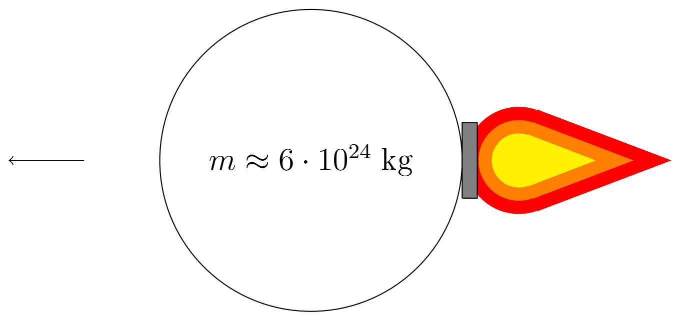
$$f(-1/2) = \ln(1/4 - 1/2 + 1) = \ln(3/4).$$

On the endpoints, we have

$$f(-1) = \ln(1 - 1 + 1) = \ln(1) = 0, \quad f(1) = \ln(1 + 1 + 1) = \ln(3).$$

Hence $f(-1/2) < f(-1) < f(1)$. The absolute minimum is $\boxed{f(-1/2) = \ln(3/4)}$, and the absolute maximum is $\boxed{f(1) = \ln(3)}$.

(6) You are designing the engine for a propulsion system which will be used to accelerate the Earth out of the solar system toward the nearest star.



Your engine must maintain a steady acceleration of $a = 0.001$ m/s^2. You want to figure out how much power P this will require. By definition, power is the rate of change of energy E over time, i.e.

$$P = \frac{dE}{dt}.$$

In other words, power consumption is how much energy you use per unit of time.

(a) You consult a physicist called Isaac Newton, who tells you the relationship between the energy E and the velocity v of an object with mass m is

$$E = \frac{1}{2}mv^2.$$

Using this formula, find a relationship between P and a.

Solution. Differentiate both sides with respect to time to get

$$\frac{dE}{dt} = mv \cdot \frac{dv}{dt}.$$

Rewrite this, using the definitions of power and acceleration, as $\boxed{P = mva}$. (Note that the mass m doesn't change with respect to time, i.e. is a constant.)

(b) The target velocity for the Earth is $v = 0.02c$, where $c \approx 3.00 \cdot 10^8$ m/s is the speed of light. What is the maximum power output required from your engine in order to maintain the acceleration $a = 0.001$ m/s^2 at this velocity?

Solution. To have acceleration $a = 0.001$ at $v = 0.02c$, we need

$$P = m \cdot (0.02c) \cdot (0.001) = \boxed{0.0002mc}$$

units of power.

(c) As the Earth passes by Jupiter, a system malfunction causes your engine to shut down. The Earth begins falling directly toward Jupiter. When the two planets are distance r apart, the Earth is falling with velocity

$$v = -\sqrt{\frac{2GM}{r}}.$$

Here $G \cong 6.67 \cdot 10^{-11}$ is the universal gravitational constant and $M \approx 1.90 \cdot 10^{27}$ kg is the mass of Jupiter. (Here the minus sign is because we are *falling.*)

(i) What is the acceleration a of the Earth as a function of r? (Hint: velocity is $v = dr/dt$, by definition.)

Solution. Remember that acceleration is the derivative of velocity *with respect to time.* So we must apply the chain rule:

$$\begin{aligned} a = \frac{dv}{dt} &= -\sqrt{2GM}\left(-\frac{1}{2}r^{-3/2} \cdot \frac{dr}{dt}\right) \\ &= -\sqrt{\frac{GM}{2r^3}} \cdot v = \boxed{-\frac{GM}{r^2}}. \end{aligned}$$

(Again, the minus sign is because we are accelerating *toward* Jupiter.)

(ii) You designed your engine to have maximum power output according to (b). At what distance r will your engine no longer be powerful enough to push the Earth away from Jupiter? (Hint: use (a).)

Solution. From (a), to create an acceleration of $a = -GM/r^2$ at velocity $v = -\sqrt{2GM/r}$, you need a power of

$$P = mva = m \cdot \frac{GM}{r^2} \cdot \sqrt{\frac{2GM}{r}} = \frac{\sqrt{2} \cdot m(GM)^{3/2}}{r^{5/2}}.$$

From (b), your engine puts out a maximum power of $P = 0.0002mc$. Solving for r, we get

$$r = \boxed{\left(\frac{(GM)^{3/2}}{0.09c}\right)^{2/5}} \approx 200000 \text{ km}.$$

Fortunately, your engine was fixed before the Earth fell below this distance.

(d) After many years, you are informed that the Earth actually has to be accelerated to $v = 0.9c$, to reach a much farther target star. You upgrade your engine to have enough power to do so, according to your formula from (a). However, as more decades pass and the Earth's velocity increases toward $v = 0.9c$, you notice that your engine seems to require more power than predicted by (a) in order to maintain $a = 0.001$ m/s. You consult another physicist called Albert Einstein, who tells you that the true relationship between E and v is

$$E = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right).$$

(Newton's formula in (a) was an approximation that was only valid for v *much* smaller than the speed of light.) Repeat parts (a) and (b) using the true formula. What is the actual maximum power output needed at $v = 0.9c$?

Solution. Differentiating both sides and remembering to use the chain rule gives

$$\begin{aligned} P &= mc^2 \cdot \left(-\frac{1}{2} \cdot \frac{1}{(1 - \frac{v^2}{c^2})^{3/2}} \right) \cdot \left(-\frac{2v}{c^2} \cdot \frac{dv}{dt} \right) \\ &= \frac{mva}{(1 - v^2/c^2)^{3/2}}. \end{aligned}$$

From (a), we would have gotten $P = mva = 0.0009mc$. Now we have to correct by a factor of

$$\frac{1}{(1 - v^2/c^2)^{3/2}} = \frac{1}{(1 - 0.9^2)^{3/2}} \approx 12,$$

i.e. we actually need $\boxed{P \approx 12 \cdot 0.0009mc}$.