MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 7 (DUE MAR 14)

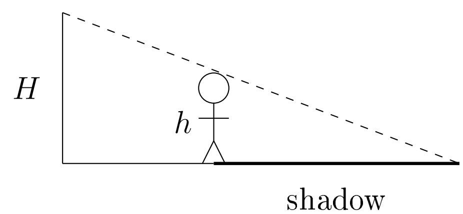
Each part (labeled by letters) of every question is worth 3 points. There are 10 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Two quantities f and g both depend on time t and are related by the equation

$$f^4 + \sin(g) = (t+1)^2.$$

Find an equation relating f' and g'. If $f(0) = 1$ and $g(0) = 0$ and $g'(0) = 6$, what is $f'(0)$?

(2) You are walking away from a streetlight and wondering how fast your shadow grows.



Let H be the height of the streetlight, and h be your own height. Write down an equation relating how far you are from the streetlight and how long your shadow is. If you are walking at 1 m/s, how fast is your shadow growing?

(3) Tired of calculus homework, you go ride a ferris wheel. A sign proclaims "biggest ferris wheel ever: height 120 meters". One minute after you get on the ride, you have rotated a quarter of the way, and you are already bored. For fun, you decide to compute the speed at which you are rising when you reach 90 meters above ground level. What is the speed? (Assume the ferris wheel rotates at a constant rate.)

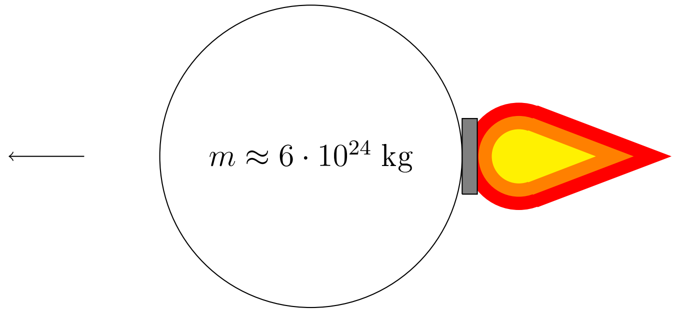
(4) Use linear approximation to estimate $\sqrt[4]{10001}$ and $\sin(29°)$.

(5) Consider the function

$$f(x) = \ln(x^2 + x + 1)$$

on the interval $[-1, 1]$. Explain why an absolute maximum must exist on this interval. Find both the absolute maximum and absolute minimum values.

(6) You are designing the engine for a propulsion system which will be used to accelerate the Earth out of the solar system toward the nearest star.



Your engine must maintain a steady acceleration of $a = 0.001$ m/s^2. You want to figure out how much power P this will require. By definition, power is the rate of change of energy E over time, i.e.

$$P = \frac{dE}{dt}.$$

In other words, power consumption is how much energy you use per unit of time.

(a) You consult a physicist called Isaac Newton, who tells you the relationship between the energy E and the velocity v of an object with mass m is

$$E = \frac{1}{2}mv^2.$$

Using this formula, find a relationship between P and a.

(b) The target velocity for the Earth is $v = 0.02c$, where $c \approx 3.00 \cdot 10^8$ m/s is the speed of light. What is the maximum power output required from your engine in order to maintain the acceleration $a = 0.001$ m/s^2 at this velocity?

(c) As the Earth passes by Jupiter, a system malfunction causes your engine to shut down. The Earth begins falling directly toward Jupiter. When the two planets are distance r apart, the Earth is falling with velocity

$$v = -\sqrt{\frac{2GM}{r}}.$$

Here $G \cong 6.67 \cdot 10^{-11}$ is the universal gravitational constant and $M \approx 1.90 \cdot 10^{27}$ kg is the mass of Jupiter. (Here the minus sign is because we are *falling*.)

(i) What is the acceleration a of the Earth as a function of r? (Hint: velocity is $v = dr/dt$, by definition.)

(ii) You designed your engine to have maximum power output according to (b). At what distance r will your engine no longer be powerful enough to push the Earth away from Jupiter? (Hint: use (a).)

Fortunately, your engine was fixed before the Earth fell below this distance.

(d) After many years, you are informed that the Earth actually has to be accelerated to $v = 0.9c$, to reach a much farther target star. You upgrade your engine to have enough power to do so, according to your formula from (a). However, as more decades pass and the Earth's velocity increases toward $v = 0.9c$, you notice that your engine seems to require more power than predicted by (a) in order to maintain $a = 0.001$ m/s. You consult another physicist called Albert Einstein, who tells you that the true relationship between E and v is

$$E = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right).$$

(Newton's formula in (a) was an approximation that was only valid for v *much* smaller than the speed of light.) Repeat parts (a) and (b) using the true formula. What is the actual maximum power output needed at $v = 0.9c$?