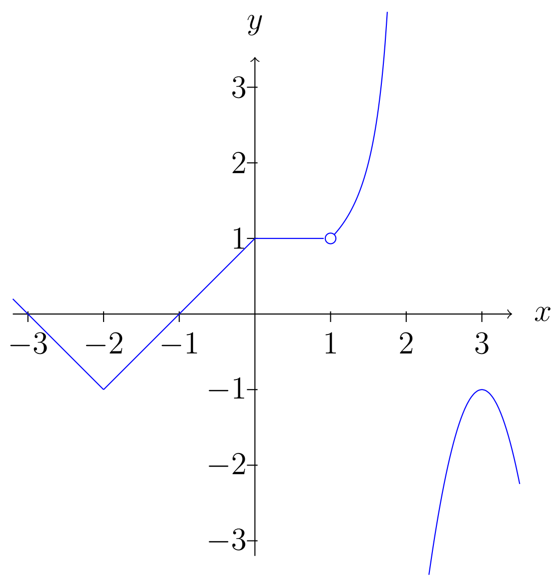
HOMEWORK 8 (DUE MAR 26)

Each part (labeled by letters) of every question is worth 2 points. There are 15 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) For each of the following functions, find its critical numbers and determine the intervals in the domain on which the function is increasing, and on which it is decreasing. Then, for each critical number, identify whether it is a local maximum or local minimum or neither.

(a) The function given by the graph



(b)

$$f(x) = \frac{x^2 - 2}{x^2 + 6}$$

(c)

$$f(x) = \ln(x^3 + 1)$$

(2) Your friend leaves your house at 11:00pm, drives to the nearest McDonalds (3 miles away) for a burger, and arrives back at 11:12pm. Prove using the mean value theorem that your friend must have exceeded the 25 miles/hr speed limit at some point during the drive.

(3) Let $f(x) = x^3 - \cos(x) + r$. Consider the equation $f(x) = 0$ on the interval $[1, \infty)$.

(a) Show that, for *any* real number r, the equation has at most one solution in the given interval.

(b) Find a value of r for which there is one solution, and another value of r for which there are no solutions. Give a rigorous justification in both cases (of the existence/non-existence of a solution).

(c) How many inflection points does f have?

(4) Find the absolute minimum and absolute maximum values of f on the given interval.

(a)

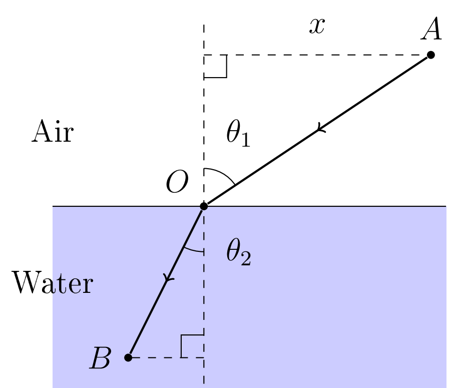
$$f(x) = \frac{x}{x^2 - x + 1}, \quad [0, 3].$$

(b)

$$f(t) = t + \cot(t/2), \quad [\pi/4, 7\pi/4].$$

(5) Consider the ellipse $x^2/a^2 + y^2/b^2 = 1$. What is the area of the largest rectangle that can be inscribed in it? (Hint: points on this ellipse are of the form $(x, y) = (a\cos\theta, b\sin\theta)$. At some point, to make your life easier, use the double angle formula $2\sin\theta\cos\theta = \sin 2\theta$.)

(6) Let v_1 be the speed of light in air, and v_2 be the speed of light in water. Because $v_1 \neq v_2$, light rays *refract* when they enter water from air.



The principle of least action says that the light ray will travel along the path which takes the *least time*.

(a) Define the following constants.

- Let ℓ be the total horizontal distance from A to B.
- Let h_1 be the height of A (above water level).
- Let h_2 be the depth of B (below water level).

Write expressions for the distances AO and OB traveled by the light ray, as functions of x (in the diagram) which involve the constants ℓ, h_1, h_2. What is the total time needed for the light ray to get from A to B?

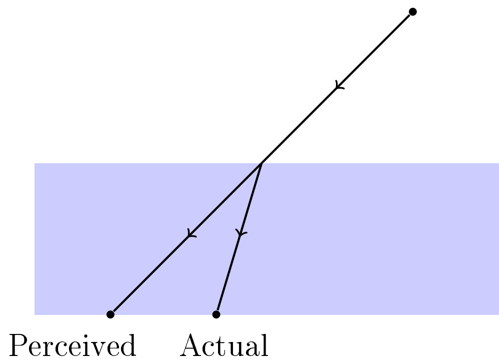
(b) The total time in (a) is a function of x. Call it $T(x)$. If $x = x_0$ is the value of x for the actual path that the light ray takes, explain why the principle of least action implies $T'(x_0) = 0$.

(c) Using (b), show that the actual path taken by the light ray must satisfy

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

This is known as Snell's law. The ratio v_1/v_2 is called the *index of refraction*; for water and air, it is approximately $4/3$.

(d) You are standing in a pond which is one meter deep. For simplicity, you are two meters tall. You look downward at a 45° angle and see a little crab crawling at the bottom of the pond. Taking refraction into account and using (c), how far is the crab from your feet? (Non-exact answers are fine.)



(7) Annoyed by your calculus homework, you crumple it into a ball and throw it into an infinitely deep hole. Because it is spring break, this action gets you a free 2 points. Go enjoy your break!