MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

PRACTICE MIDTERM 1 SOLUTIONS

The exam is **75 minutes**. No additional material or calculators are allowed.

- Write your **name and UNI** clearly on your exam booklet.
- **Show your work** and reasoning, not just the final answer. Partial credit will be given for correct reasoning, even if the final answer is completely wrong.
- **Don't cheat**!
- Don't panic!

(1) (10 points) State whether the following are true/false. No explanations necessary.

(a) For every real number x, the identity $\sin(\sin^{-1}(x)) = x$ holds.

False. Numbers x outside of $[-1, 1]$ are not even in the domain of $\sin^{-1}$, so the left hand side is not even defined for all real numbers.

(b) If $f(x)$ is continuous at $x = a$, then $f(x)$ is differentiable at $x = a$.

False. The absolute value function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there.

(c) The function $f(x) = xe^{\sin(x)}$ is continuous on its domain.

True. It is the composition and product of the continuous functions x, e^x, and $\sin(x)$.

(d) The limit

$$\lim_{x\to 0} \frac{x\sin(4/x^3)}{\sqrt{x}}$$

exists.

True. First simplify the expression into $\sqrt{x}\sin(4/x^3)$. Then note that

$$-\sqrt{x} \leq \sqrt{x}\sin(4/x^3) \leq \sqrt{x}.$$

Since $\lim_{x\to 0} \pm\sqrt{x} = 0$, by squeeze theorem the desired limit is also 0. In particular it exists.

(e) If f is continuous at 2, then $\lim_{x\to 2} f(x + 4)$ must exist.

False. The existence of the limit $\lim_{x\to 2} f(x+4)$ requires that $f(x)$ is continuous at 6, not 2. For example, if f has a jump discontinuity at $x = 6$, the limit cannot exist.

(f) The equation $x^7 + x - 1 = 0$ has a solution in $(0, 1)$.

True. Let $f(x) = x^7 + x - 1$. Note that $f(0) = -1 < 0$ and $f(1) = 1 > 0$, so by intermediate value theorem there is some value $c \in (0, 1)$ such that $f(c) = 0$, i.e. c is a solution.

(g) If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x)/g(x)$ does not exist.

False. Take $f(x) = g(x) = x - a$.

(h) $f(a) = g(a)$ for all real numbers a, where

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = x + 2.$$

False. $f(2)$ is undefined, but $g(2) = 4$.

(i) $\lim_{x \to a} f(x) = g(a)$ for all real numbers a, where

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = x + 2.$$

True. Clearly $\lim_{x \to a} f(x) = f(a) = g(a)$ for $a \neq 2$. When $a = 2$, we have $\lim_{x \to 2} f(x) = 2 + 2 = 4 = g(2)$.

(j) The derivative of $f(x) = (2x)^3$ is $3 \cdot (2x)^2 = 12x^2$.

False. Remember that the power rule only applies to things of the form x^n. To differentiate $(2x)^3$, first pull out the constant 2^3, to get the derivative $2^3 \cdot (3x^2) = 24x^2$.

(2) Find the limit, if it exists. If it does not exist, explain why.

(a) (5 points)

$$\lim_{x\to 0} f(|x|),$$

where the function $f(x)$ is

$$f(x) = \begin{cases} \ln(-x) & x < 0 \\ \cos^{-1}(x) & x \geq 0 \end{cases}.$$

Solution. Since $f(|x|)$ is secretly a piecewise function, let's check whether left and right limits exist and are equal.

(i) When $x \to 0^-$, we have $|x| = -x$. This quantity is always positive, so $f(-x) = \cos^{-1}(-x)$. By continuity, as $x \to 0$, the limit is $\cos^{-1}(0) = \pi/2$.

(ii) When $x \to 0^+$, we have $|x| = x$. Again, this quantity is always positive, so again $f(x) = \cos^{-1}(x)$, whose limit is again $\pi/2$.

It follows that $\boxed{\lim_{x\to 0} f(|x|) = \pi/2}$.

(b) (5 points)

$$\lim_{x\to\infty} \frac{e^{-x}}{6+\cos(x)}.$$

Solution. There is a $\cos(x)$ here, so try using squeeze theorem. We have

$$\frac{e^{-x}}{6+1} \leq \frac{e^{-x}}{6+\cos(x)} \leq \frac{e^{-x}}{6-1}.$$

Since $\lim_{x\to\infty} e^{-x} = 0$, the same is true of $e^{-x}/7$ or $e^{-x}/5$. So both lower and upper bounds have limit 0. By squeeze theorem, the desired limit is also $\boxed{0}$.

(3) Consider the function

$$f(x) = \frac{g(x)+1}{g(x)-1}.$$

(a) (5 points) If $g(x)$ is continuous everywhere except $x = 2$, where is $f(x)$ continuous?

Solution. In addition to where $g(x)$ fails to be continuous, $f(x)$ also needs $g(x) \neq 1$. So f is continuous on $\boxed{\{x : x \neq 2 \text{ and } g(x) \neq 1\}}$.

(b) (5 points) State what it means for $f(x)$ to have a horizontal asymptote $y = L$. If $\lim_{x\to\pm\infty} g(x) = \infty$, what must L be?

Solution. We say $f(x)$ has a horizontal asymptote $y = L$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$. If $g(x)$ blows up, then

$$\lim_{x\to\infty} \frac{g(x)+1}{g(x)-1} = \lim_{x\to\infty} \frac{g(x)}{g(x)} = 1.$$

This is because $g(x)$ grows much faster than the constants ± 1. Equivalently,

$$\lim_{x\to\infty} \frac{g(x)+1}{g(x)-1} = \lim_{x\to\infty} \frac{1+\frac{1}{g(x)}}{1-\frac{1}{g(x)}} = \lim_{x\to\infty} \frac{1}{1} = 1,$$

since $\lim_{x\to\infty} 1/g(x) = 0$. Hence $\boxed{L = 1}$.

(c) (5 points) Compute the derivative of $f(x)$. State important differentiation rules that you use.

Solution. Since $f(x)$ is a quotient, we directly apply the quotient rule to get

$$f'(x) = \frac{g'(x)\cdot(g(x)-1) - (g(x)+1)\cdot g'(x)}{(g(x)-1)^2} = \boxed{\frac{-2g'(x)}{(g(x)-1)^2}}.$$

(4) Let $f(x) = A/(3-x)$ for some constant A.

(a) (5 points) Write down the definition of the derivative $f'(x)$, and compute the derivative $f'(1)$ using the definition.

Solution. The definition says

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{A}{3-(x+h)} - \frac{A}{3-x}}{h}.$$

We want $f'(1)$, so plug in $x = 1$ and take the limit:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{\frac{A}{2-h} - \frac{A}{2}}{h} \\ &= \lim_{h\to 0} \frac{2A - A(2-h)}{2h(2-h)} \\ &= \lim_{h\to 0} \frac{A}{2(2-h)} = \boxed{\frac{A}{4}}. \end{aligned}$$

(b) (5 points) If the tangent line to $f(x)$ at $x = 1$ is $y = 2x + 2$, what is A?

Solution. The slope of the tangent line to $f(x)$ at $x = 1$ is, by definition, $f'(1)$. We computed this to be $A/4$. The line $y = 2x + 2$ has slope 2, so it follows that $A/4 = 2$ and $\boxed{A = 8}$.

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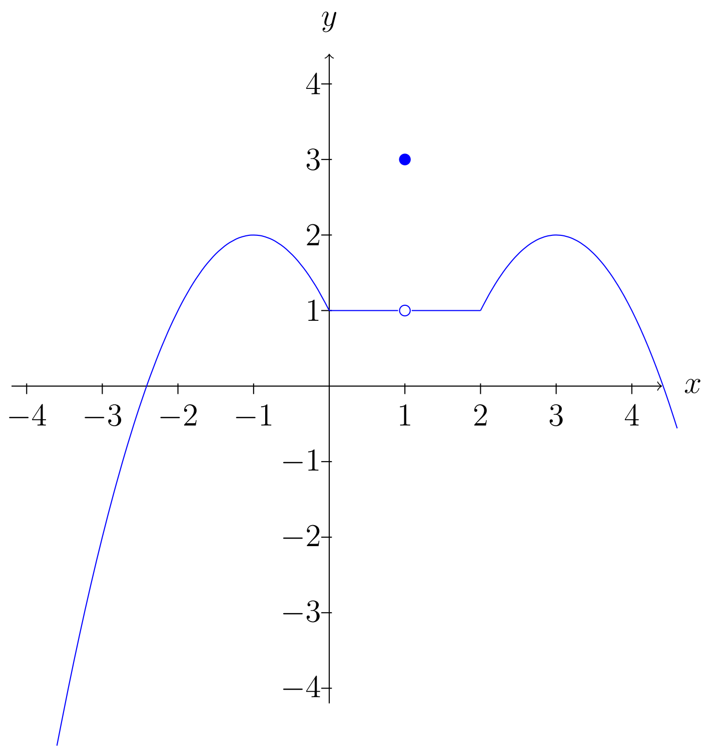
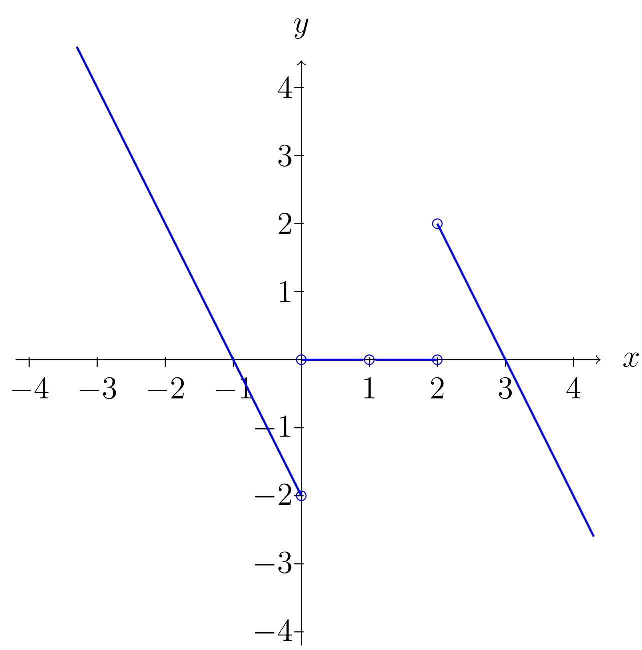
(5) (5 points) For the following graph, state where it fails to be continuous and where it fails to be differentiable. Where the derivative exists, sketch its derivative.



Solution. The function is $\boxed{\text{continuous everywhere except } x = 1}$. So it automatically fails to be differentiable at $x = 1$. It also fails at $x = 0$ and $x = 2$, because the slope abruptly changes. So it is $\boxed{\text{differentiable everywhere except } x = 0, 1, 2}$. The derivative looks like:



To roughly sketch the derivative, note that the slope is 0 at $x = -1$ and $x = 3$. So $f'(-1) = f'(3) = 0$.
(a) On $(-\infty, -1)$, the slope is positive.
(b) On $(-1, 0)$, the slope is negative.
(c) On $(0, 1)$ and $(1, 2)$, the slope is zero.
(d) On $(2, 3)$, the slope is positive.
(e) On $(3, \infty)$, the slope is negative.
(It doesn't matter whether you draw straight or curved lines.)