MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

# PRACTICE MIDTERM 2 SOLUTIONS

The exam is **75 minutes**. No additional material or calculators are allowed.

- Write your **name and UNI** clearly on your exam booklet.
- **Show your work** and reasoning, not just the final answer. Partial credit will be given for correct reasoning, even if the final answer is completely wrong.
- **Don't cheat**!
- Don't panic!

(1) (10 points) State whether the following are true/false. No explanations necessary.

(a) Since $\sec x = (\cos x)^{-1}$, the derivative of $\sec x$ is $-(\cos x)^{-2}$.

**Solution.** $\boxed{\text{False}}$. Don't forget the chain rule. The derivative is actually

$$\frac{d}{dx}(\sec x) = -(\cos x)^{-2} \cdot (-\sin x).$$

(b) There exists a differentiable function $f(x)$ such that $f'(x) < 1$ and $f(0) = 0$ and $f(2) = 2$.

**Solution.** $\boxed{\text{False}}$. By the mean value theorem, $f(2) - f(0) = f'(c)(2 - 0) < 2$, so $f(2) < 2$. Intuitively, if the slope of $f$ is always less than 1, it must grow more slowly than $f(x) = x$, and therefore must be $< 2$ at $x = 2$.

(c) The function $\tan(x)$ has an absolute maximum on $[0, \pi/2)$.

**Solution.** $\boxed{\text{False}}$. There is a vertical asymptote as $x \to \pi/2$, and so an absolute maximum is never achieved.

(d) The function $\tan(x)$ has an absolute minimum on $[0, \pi/2)$.

**Solution.** $\boxed{\text{True}}$. The absolute minimum is at $\tan(0) = 0$.

(e) If a function $f$ is continuous on an interval $[a, b]$, it must have a critical value in $(a, b)$.

**Solution.** $\boxed{\text{False}}$. For example, $f(x) = x$ has no critical values at all, on any interval. This example does not contradict the extreme value theorem: the extreme values will always be on the endpoints.

(f) There exists $c$ in the interval $(1, 2)$ such that the function

$$f(x) = x^3 - x + \cos(\pi/x)$$

has derivative $f'(c) = 7$.

**Solution.** $\boxed{\text{True}}$. Use mean value theorem: $f(1) = -1$ and $f(2) = 6$, so there must exist $c$ in $(1, 2)$ such that $f'(c)$ is equal to the average slope $(6-(-1))/(2-1) = 7$.

(g) Let $g(x)$ be the inverse function of $f(x) = xe^x$ (e.g. $f(g(x)) = x$). Then $g'(0) = 2$.

**Solution.** $\boxed{\text{False}}$. From a previous homework, or via the chain rule, the derivative of the inverse function $g(x)$ may be calculated as $g'(x) = 1/f'(g(x))$. We want to compute $g'(0) = 1/f'(g(0))$. Since $f(0) = 0$, it follows that $g(0) = 0$. So

$$g'(0) = \frac{1}{f'(g(0))} = \frac{1}{f'(0)} = \frac{1}{e^0 + 0 \cdot e^0} = 1.$$

(h) $\lim_{x\to 0} x^{\sqrt{x}}$ does not exist.

**Solution.** $\boxed{\text{False}}$. First rewrite the limit as

$$\lim_{x\to 0} x^{\sqrt{x}} = \exp\left(\lim_{x\to 0} \sqrt{x} \cdot \ln(x)\right).$$

This is an indeterminate form, so we first rewrite it as a fraction, and then apply l'Hôpital's rule:

$$\lim_{x\to 0} \sqrt{x} \cdot \ln(x) = \lim_{x\to 0} \frac{\ln(x)}{x^{-1/2}} = \lim_{x\to 0} \frac{1/x}{-(1/2)x^{-3/2}} = -2 \lim_{x\to 0} x^{1/2} = 0.$$

(i) If $f'(c) = 0$, then $f(c)$ is either a local maximum or a local minimum.

**Solution.** $\boxed{\text{False}}$. For example, $f(x) = x^3$ satisfies $f'(0) = 0$, but $x = 0$ is neither a local max nor local min.

(j) There exists a function $f$ such that $f(x) > 0$ and $f'(x) < 0$ and $f''(x) > 0$ for all $x$.

**Solution.** $\boxed{\text{True}}$. The condition $f'(x) < 0$ means the slope is always negative, and $f''(x) > 0$ means concave up. It is fairly straightforward to draw such a function: it looks like $f(x) = e^{-x}$.

(2) Compute the derivative $dy/dx$. Write it as a function of just $x$ if possible.

(a) (5 points)

$$y = \frac{(x+1)^5(x-2)^6}{\sqrt{2x-5}}$$

(Hint: take ln of both sides.)

**Solution.** The hint tells us to use logarithmic differentiation. So first ln both sides:

$$\ln y = 5\ln(x+1) + 6\ln(x-2) - \frac{1}{2}\ln(2x-5).$$

Now differentiate both sides, keeping in mind that $y$ depends on $x$:

$$\frac{1}{y} \cdot y' = \frac{5}{x+1} + \frac{6}{x-2} - \frac{1}{2} \cdot \frac{1}{2x-5} \cdot 2.$$

Move the $y$ to the right hand side, and substitute in the expression for $y$, to get:

$$\boxed{y' = \frac{(x+1)^5(x-2)^6}{\sqrt{2x-5}}\left(\frac{5}{x+1} + \frac{6}{x-2} - \frac{1}{2x-5}\right)}.$$

(b) (5 points)

$$e^{xy} - y = x.$$

**Solution.** Use implicit differentiation:

$$e^{xy} \cdot (1 \cdot y + x \cdot y') - y' = 1.$$

Solve for $y'$ and rearrange to get

$$\boxed{y' = \frac{1 - ye^{xy}}{xe^{xy} - 1}}.$$

(In principle, we can simplify this further by substituting $x + y = e^{xy}$, but I'm happy if you leave it in this form.)

(3) (5 points) Use linear approximation to give an estimate for $\tan(\pi/180)$. (Leave $\pi/180$ alone; no need to calculate its actual value.) If you repeated the same procedure to estimate $\tan(\pi/90)$, would it be more or less accurate than your estimate for $\tan(\pi/90)$? Briefly explain.

**Solution.** Since the derivative of $\tan(x)$ is $\sec^2(x)$, linear approximation says that for $x$ close to 0,

$$\tan(x) \approx \tan(0) + \sec^2(0) \cdot x.$$

We know $\tan(0) = 0$ and $\sec^2(0) = 1/\cos^2(0) = 1$. So

$$\boxed{\tan(\pi/180) \approx \pi/180}.$$

The estimate would be $\boxed{\text{less accurate}}$ for $x = \pi/90$, because the tangent line approximates the function more and more closely as $x$ approaches the point of tangency. So as we move away from the point of tangency, it becomes less and less accurate in general.

(4) Let $f(x) = x^4 - 4x^3 + 4x^2 - 1$.

(a) (5 points) Find the critical points. For each, determine whether it is a local minimum, local maximum, or neither.

**Solution.** Critical points are determined by setting

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x^2 + 2) = 4x(x-1)(x-2).$$

to zero and solving for $x$. So the critical points are $\boxed{x = 0, 1, 2}$. Use the second derivative test to determine whether these are local maxs/mins. We have

$$f''(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2).$$

(i) Since $f''(0) = 4 \cdot 2 > 0$, $\boxed{x = 0 \text{ is a local min}}$.
(ii) Since $f''(1) = 4 \cdot -1 < 0$, $\boxed{x = 1 \text{ is a local max}}$.
(iii) Since $f''(2) = 4 \cdot 2 > 0$, $\boxed{x = 2 \text{ is a local min}}$.

(b) (5 points) Find the inflection points, and the intervals where $f$ is concave up and concave down.

**Solution.** Inflection points are where $f''(x) = 0$. Solve this using the quadratic formula to get the inflection points

$$x = \frac{6 \pm \sqrt{6^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = \boxed{1 \pm \frac{\sqrt{3}}{3}}.$$

So we need to see what happens on the intervals $(-\infty, 1 - \sqrt{3}/3)$ and $(1 - \sqrt{3}/3, 1 + \sqrt{3}/3)$ and $(1 + \sqrt{3}/3, \infty)$.

(i) Since $f''(-10000) > 0$, $\boxed{f \text{ is concave up on } (-\infty, 1 - \sqrt{3}/3)}$.

(ii) Since $f''(1) < 0$, $\boxed{f \text{ is concave down on } (1 - \sqrt{3}/3, 1 + \sqrt{3}/3)}$.

(iii) Since $f''(10000) > 0$, $\boxed{f \text{ is concave up on } (1 + \sqrt{3}/3, \infty)}$.

(c) (3 points) What is the absolute maximum and absolute minimum on the interval $[-1, 4]$?

**Solution.** We already found the critical points $x = 0, 1, 2$, with values

$$f(0) = -1, \quad f(1) = 0, \quad f(2) = 16 - 32 + 16 - 1 = -1.$$

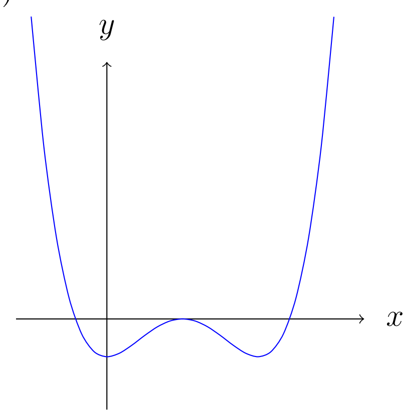
On the endpoints, we have

$$f(-1) = 8, \quad f(4) = 4^4 - 4^4 + 4^3 - 1 = 63.$$

Hence the $\boxed{\text{absolute max is } 63}$ and the $\boxed{\text{absolute min is } -1}$.

(d) (2 points) Roughly sketch the graph.

**Solution.** (Not to scale)




(5) (5 points) The area of an equilateral triangle is growing at 30 $\text{cm}^2/\text{min}$. How fast are the sides growing when they are exactly $\sqrt{3}$ cm?

**Solution.** Let $A$ be the area of the triangle, and $s$ be the side length. We are told $dA/dt = 30$ and are asked for $ds/dt$ when $s = \sqrt{3}$. This is a related rates problem. The relationship between $A$ and $s$ is

$$A = \frac{1}{2} \cdot (\text{base}) \cdot (\text{height}) = \frac{1}{2} \cdot s \cdot \frac{\sqrt{3}}{2} s.$$

Differentiating,

$$\frac{dA}{dt} = \frac{1}{2}\frac{\sqrt{3}}{2} \cdot 2s \cdot \frac{ds}{dt} = \frac{\sqrt{3}}{2} \cdot s\frac{ds}{dt}.$$

Hence when $s = \sqrt{3}$, we get

$$\frac{ds}{dt} = \frac{2}{\sqrt{3} \cdot s} \cdot \frac{dA}{dt} = \frac{2}{3} \cdot 30 = \boxed{20} \text{ cm/min}.$$

(6) (5 points) Prove that among all rectangles with the same area $A$, the one with smallest perimeter is a square. (Hint: let the side lengths be $x$ and $y$, and minimize the perimeter.)

**Solution.** Use the hint. The area is $A = xy$ and the perimeter is $P = 2x + 2y$. Substitute $y = A/x$ into this to get the perimeter as a function of just $x$:

$$P(x) = 2x + \frac{2A}{x}.$$

We want to find the absolute minimum for the perimeter. Solve

$$0 = P'(x) = 2 - 2Ax^{-2}$$

to get that $x = \sqrt{A}$. So $y = A/x = \sqrt{A}$ as well, i.e. both sides are the same length, forming a square.

(For completeness, you should check that this is a local *minimum* instead of a local *maximum*. One fast way to do this is via the first derivative test: if $x < \sqrt{A}$, then $P'(x) < 0$, and if $x > \sqrt{A}$, then $P'(x) > 0$. So $x = \sqrt{A}$ is indeed a local min.)