Today: prove Nekrasov's formula.

$$\chi\left(\mathrm{Hilb}(\mathbb{C}^2),\ \Lambda^\bullet_{-m}(T^\vee_{\mathrm{Hilb}}) \cdot Q^{\deg}\right) = S^\bullet\left(\underbrace{\chi\left(\mathbb{C}^2, \Lambda^\bullet_{-m}(T^\vee_{\mathbb{C}^2})\right)}_{\frac{(1-m/t_1)(1-m/t_2)}{(1-1/t_1)(1-1/t_2)}} \frac{Q}{1-mQ}\right)$$

χ_y-genus

$T = (\mathbb{C}^\times)^2$ with weights t_1, t_2

$\Lambda^\bullet_{-m}(T^\vee_{\mathrm{Hilb}}) = \sum_i (-m)^i \Omega^i_{\mathrm{Hilb}}$

Here $S^\bullet(f) = \exp\left(\sum_{m>0} \frac{1}{m} \tau^m(f)\right)$ is the "plethystic exponential", generalizing symmetric algebra $\mathrm{Sym}^\bullet(V)$

f: function of equiv. variable

τ^m: Adams operation $t_i \mapsto t_i^m$

e.g. $S^\bullet(t) = \exp\left(\sum_{m>0} \frac{1}{m} t^m\right) = \exp\left(-\log(1-t)\right) = \frac{1}{1-t}$

t: character of V; $\frac{1}{1-t}$: character of $\mathrm{Sym}^\bullet V$

$S^\bullet(f+g) = S^\bullet(f) \cdot S^\bullet(g) \Rightarrow S^\bullet(-t) = \frac{1}{S^\bullet(t)} = 1-t = \Lambda^\bullet_{-1}(t)$

not a coincidence

$\cdots \to \Lambda^1 V \otimes S V \to S^\bullet(V) \to \mathbb{C} \to 0$

Koszul complex.

Note: rhs has the form

$$S^\bullet \sum_{n>0} Q^n g_n = \sum_{m_1, m_2, \dots \geq 0} Q^{\sum k m_k} \prod_k S^{m_k}(g_k)$$

picking from $\mathbb{Z}_{>0}$ with replacement.
$|||$
picking m_k copies of $k \in \mathbb{Z}_{>0}$
$|||$
a partition μ.

while lhs is $\sum_{n>0} Q^n f_n$

$$\begin{pmatrix} f_n = \chi(\mathrm{Hilb}_n, \mathcal{F}_n) \\ g_n = \chi(\mathbb{C}^2, \mathcal{G}_n) \end{pmatrix}$$

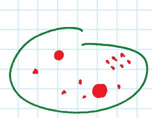
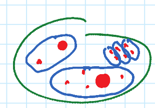
When will $\chi(\mathcal{F}_n) = \sum \prod S^{m_k} \chi(\mathcal{G}_n)$?

Idea: $\mathrm{Hilb}(\mathbb{C}^2) \xrightarrow{\pi} \mathrm{Sym}(\mathbb{C}^2) = \bigsqcup (\mathbb{C}^2)^n / S_n$

sheaves $\bigsqcup_n \mathcal{F}_n = \mathcal{F}$ are sometimes factorizable, namely

$$\mathcal{F}_n\big|_U = \boxtimes_k S^{m_k} \mathcal{F}_k$$

$\cap$ locus where points in different groups are different, e.g. $\{\{p, q\}, \{r\}\}$

$\left[\mathrm{Sym}^n(\mathbb{C}^2) \ni\right.$  $\left.\mathbb{C}^2\right] \supset \left[\prod_k \mathrm{Sym}^{m_k} \mathrm{Sym}^k \mathbb{C}^2 \ni\right.$  $\left.\mathbb{C}^2\right]$

$\exists\, m_k$ groups of size k

e.g. $\pi_* T_{\mathrm{Hilb}(\mathbb{C}^2)}$ is factorizable (exercise).

$\pi_* \Omega^i_{\mathrm{Hilb}(\mathbb{C}^2)}$ is too.

Lemma: Factorizable $\Rightarrow$ $\exists \{\mathcal{G}_n\}_{n>0}$ on $\mathbb{C}^2$ s.t.

$$\mathcal{F}_n = \bigoplus_{\sum k m_k = n} \prod_k S^{m_k} \mathcal{G}_k$$

a geometric form of inclusion-exclusion.

Pf: Construct $\{\mathcal{G}_n\}$ inductively.

$$\mathcal{F}_1 =: \mathcal{G}_1$$

$$\mathcal{F}_2 = S^2 \mathcal{F}_1 + \mathcal{G}_2$$

lives on Δ by excision. $\to K_T(\Delta) \to K_T(S^2\mathbb{C}^2) \to K_T(\mathbb{C}^2 \times \mathbb{C}^2 \setminus \Delta / S_2) \to 0$

locus where 2 pts collide.

$\Delta \subset \mathbb{C}^2 \times \mathbb{C}^2$

$\mathcal{F}_2|_{(\mathbb{C}^2\times\mathbb{C}^2)\setminus\Delta / S_2} = S^2 \mathcal{F}_1$ by factorization

In general, stratify:

$$S^n\mathbb{C}^2 = Y_n \supset Y_{n-1} \supset \cdots \supset Y_1 \simeq \mathbb{C}^2$$

get excess contribution when all pts collide: this is $\mathcal{G}_n$.

so each $\displaystyle Y_{n-k} \setminus Y_{n-k-1} = \bigcup_{\mu \text{ of exactly } k \text{ parts.}}$ (locus where factorization applies.)

$\Rightarrow$ set $\mathcal{G}_n$ to be remaining contribution supported only on $Y_1 \simeq \mathbb{C}^2$ $\square$.

$$\Rightarrow \quad \chi\left(\mathrm{Hilb},\ \Omega^\bullet_{-m} \cdot Q^{\dim}\right) = S^\bullet \chi\left(\mathbb{C}^2, \mathcal{G}(Q)\right) = S^\bullet\left(\frac{1}{(1-t_1^{-1})(1-t_2^{-1})}\, g(Q)\right)$$

$\mathcal{G}(Q) = \sum_k Q^k \mathcal{G}_k$

$\mathcal{G}(Q)|_0 \in K_T(pt)[m^{\pm}][[Q]]$

Now use some tricks to compute $g(Q)$:

1. Denominator $(1-t_1^{-1})(1-t_2^{-1})$ comes from $T_Z \mathrm{Hilb} = t_1 + t_2 + \cdots$ (exercise)

   $\Rightarrow$ $\Omega^\bullet_{-m} = \Lambda^\bullet_{-m} T^\vee$ creates a numerator $(1-mt_1^{-1})(1-mt_2^{-1})$

2. LHS is rigid $\Rightarrow$ compute $$S^\bullet\left(\underbrace{\frac{(1-mt_1^{-1})(1-mt_2^{-1})}{(1-t_1^{-1})(1-t_2^{-1})}}_{\text{all limits in } t_i\text{'s exist.}} \cdot\ ??\right)$$

   all limits in equivariant variables exist.

   ?? has to be independent of t_i's.

   by taking any limit in t_i's.

$\mathbb{C}^2$ is symplectic $\Rightarrow$ $\mathrm{Hilb}(\mathbb{C}^2)$ is too.
($\omega = dx \wedge dy$)

e.g. imagine $T_0\mathbb{C}^2 = t_1 + (t_1t_2)\cdot t_1^{-1}$

can split $T_{Z_\lambda} = T_{Z_\lambda}^{1/2} + (t_1t_2)(T_{Z_\lambda}^{1/2})^\vee$

a Lagrangian subspace

$$\mathrm{LHS} = \sum_\lambda Q^{|\lambda|} \underbrace{\prod_{w \in T_{Z_\lambda}} \frac{1-mw^{-1}}{1-w^{-1}}}$$

$$\prod_{w \in T_{Z_\lambda}^{1/2}} \frac{1-mw^{-1}}{1-w^{-1}} \cdot \frac{1-mt_1t_2w}{1-t_1t_2w}$$

$$\prod_{w \in T^{1/2}_{z_\lambda}} \frac{1-mw^{-1}}{1-w^{-1}} \frac{1-mt_1t_2w}{1-t_1t_2w}$$

becomes 1 or m   becomes m or 1   assuming $w \neq t^k$

equivalently $t_1 \to \infty$, $t_2 \to 0$, $|t_1| \gg |t_2^{-1}|$

Take limit $t_1 \to \infty$, $t_1t_2 =: t$ constant. $= m^{\#\{w \neq t^k\}} \prod \frac{1-mt^{-k}}{1-t^{-k}} \frac{1-mt^{k+1}}{1-t^{k+1}}$

then $t \to \infty$ $= m^{\operatorname{rk} T^{1/2}_{z_\lambda}} = m^{|\lambda|}$

fact: $\dim \mathrm{Hilb}_n(\mathbb{C}^2) = 2n$.

$\Rightarrow \quad \sum_\lambda Q^{|\lambda|} m^{|\lambda|}$ is the LHS in this limit.

$$\sum_\lambda Q^{|\lambda|} m^{|\lambda|} = \sum_\lambda (Qm)^{|\lambda|} \overset{\text{from factorization}}{=} S^\bullet(m \cdot \, ??)$$

$$\sum_\lambda (Qm)^{|\lambda|} = \prod_{k>0} \frac{1}{1-(Qm)^k} = S^\bullet\left(Qm + (Qm)^2 + (Qm)^3 + \cdots\right)$$

$$\Rightarrow \quad ?? = \frac{Q}{1-mQ}.$$