Equivariant K-theory: Lecture 11

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4:38 PM

Symmetric functions & $R(S_n)$

Let $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ ← homogeneous deg-n symmetric functions in variables $x_1, x_2, x_3, \ldots$

Many bases indexed by partitions λ:

e.g. power sums (over $\mathbb{Q}$) $\quad p_r := \sum_i x_i^r \qquad p_\lambda := \prod_i p_{\lambda_i}$ ← a basis for Λ as a $\mathbb{Q}$-module.

monomial: $\quad m_\lambda := (x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \cdots) + \text{all permutations.}$

elementary:
$$e_r := \sum_{i_1 > i_2 > \cdots > i_r} x_{i_1} x_{i_2} \cdots x_{i_r} = [t^r] \prod_i (1 + x_i t) \qquad e_\lambda := \prod_i e_{\lambda_i}$$

homogeneous
$$h_r := \sum_{i_1 \geq i_2 \geq \cdots \geq i_r} x_{i_1} x_{i_2} \cdots x_{i_r} = [t^r] \prod_i \frac{1}{1 - x_i t} \qquad h_\lambda := \prod_i h_{\lambda_i}$$

$\exists$ a Hall inner product:

$$\langle p_\lambda, p_\mu \rangle_\Lambda = \delta_{\lambda\mu} z_\lambda$$ ← some combinatorial factor.

equivalently:
$$\langle h_\lambda, m_\mu \rangle_\Lambda = \delta_{\lambda\mu}$$

Let $R = \bigoplus_n R_n$ ← rep. ring of S_n

has product $\underset{S_n}{V} \cdot \underset{S_m}{W} := \operatorname{Ind}_{S_n \times S_m}^{S_{n+m}} \underset{S_n \times S_m}{V \otimes W}$

Irreps in R are indexed by partitions. & in bijection with their characters:

$$\chi_V : S_n \to \mathbb{C}$$
$$\sigma \mapsto \operatorname{tr}_V \sigma$$

are class functions: $\chi_V(ghg^{-1}) = \chi_V(h)$

conjugacy classes = indexed by partitions μ

$$\frac{1}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(\mu)$$ ← value of χ_λ on conjugacy class of μ.

↑ from earlier

from earlier

$\exists$ an inner product (on class functions)

$$\langle f, g \rangle_R := \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) \overline{g(\sigma)}$$

Satisfy: $\langle \chi_V, \chi_W \rangle_R \underset{\text{(exercise.)}}{=} \dim \operatorname{Hom}_{S_n}(V, W) \in \mathbb{Z}_{\geq 0}$

important positivity.

$\Rightarrow$ in particular, irreps are orthogonal.

Thm:

$$(R, \langle -,- \rangle_R) \xrightarrow[F]{\sim} (\Lambda, \langle -,- \rangle_\Lambda)$$

iso of graded rings w/ inner product.

$$V \longmapsto \sum_\mu \frac{1}{z_\mu} \chi_V(\mu)\, p_\mu$$

"Frobenius characteristic"

(exercise.)

Eg: Kostka numbers $K_{\lambda,\mu} := \langle s_\lambda, h_\mu \rangle_\Lambda$ form a change-of-basis

Schur functions (some other basis of Λ)

$$s_\lambda = \sum_\mu K_{\lambda,\mu}\, m_\mu \qquad \in \Lambda$$

In R, have:

$$F\left(\begin{smallmatrix}\text{trivial rep}\\ \text{of } S_n\end{smallmatrix}\ 1_n\right) = \sum_{\mu \vdash n} \frac{1}{z_\mu} p_\mu \underset{}{\overset{\text{(exercise)}}{=}} h_n$$

$$F\left(\begin{smallmatrix}\text{irrep}\\ \text{rep}\end{smallmatrix}\ V_\lambda\right) \underset{\text{(exercise or the definition)}}{=} s_\lambda$$

$$\Rightarrow \quad K_{\lambda,\mu} = \langle V_\lambda, \underbrace{1_{\mu_1} \otimes 1_{\mu_2} \otimes \cdots}_{\text{induced up.}} \rangle_R = \dim \operatorname{Hom}(\cdots) \geq 0.$$

a nontrivial positivity result!

Note: such changes of basis are often upper triangular wrt. "dominance ordering" $\mu \leq \lambda$, i.e.

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu}\, m_\mu$$

along with $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$, is an over-characterization along with $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$, is an over-characterization of s_λ (by Gram-Schmidt).

(q,t) - symmetric functions :

Consider bi-graded S_n-modules $V = \bigoplus_{r,s\in\mathbb{Z}} V_{r,s}$

$\Rightarrow$ Frobenius characteristic generalizes to:

$$F_{q,t} : R_{\text{bi-graded}} \longrightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}(q,t) =: \Lambda_{q,t}$$

$$V \longmapsto \sum_{r,s} F(V_{r,s})\, q^r t^s$$

same as R, but with bigraded modules.

Some "K-theoretic operations" on $\underbrace{\mathbb{Z}[q^{\pm}, t^{\pm}]}_{\substack{K_T(pt) \\ T=(\mathbb{C}^\times)^2 \text{ with weights } q,t.}}$ lift to $\Lambda_{q,t}$:

e.g. $(\star)$ $p_k[A] := A\Big|_{\substack{q,t \mapsto q^k, t^k \\ x_i \mapsto x_i^k}}$ $\leftarrow$ Adams operation in K-th. $\qquad A \in \Lambda_{q,t}$

Since $\Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots]$, each A defines

$$\left(f \longmapsto f[A]\right) \in \mathrm{End}(\Lambda_{q,t})$$

plethystic substitution. $\qquad$ ↑ expand f in $\{p_k\}$ and apply $(\star)$

Let $X := \sum_i x_i$, so e.g. $p_k[X] = p_k \Rightarrow f[X] = f$

e.g. $s_2[X(1-t)] = \dfrac{p_{1,1} + p_2}{2} [X(1-t)]$

$$= \frac{p_{1,1}(1-t)^2}{2} + \frac{p_2(1-t^2)}{2} = (1-t)s_2 - t(1-t)s_{1,1}$$

e.g. $S^\bullet(f) = \exp\left(\sum_{k>0} p_k[f]/k\right)$.

Macdonald polynomials (Haiman's normalization)

$\{\tilde{H}_\lambda\} \subset \Lambda_{q,t}$ (over) characterized by:

$$\tilde{H}_\lambda[X(1-q)] \in \mathbb{Q}(q,t)\{s_\mu : \mu \geq \lambda\}$$

$$\tilde{H}_\lambda[X(1-t)] \in \mathbb{Q}(q,t)\{s_\mu : \mu \geq \lambda^t\}$$

$$\langle s_{(n)}, \tilde{H}_\lambda \rangle_{\Lambda_{q,t}} = 1 \qquad |\lambda| = n$$

$$\langle f, g \rangle_{\Lambda_{q,t}} = \langle f, g[X\tfrac{1-q}{1-t}] \rangle_\Lambda$$

Unify many previous families of sym. functions:

$q \to 0$: Hall-Littlewood polys.

$t = q^\alpha$ $q \to 1$: Jack polys ($\alpha \to 0$: Schur)

Macdonald positivity conjecture (now a thm.):

(q,t)-Kostka numbers.

$$\tilde{H}_\lambda = \sum_\mu \tilde{K}_{\lambda,\mu} \, s_\mu$$

for $\tilde{K}_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}[q^{\pm}, t^{\pm}] \subset \mathbb{Q}(q,t)$

Plan of attack: realize $\tilde{H}_\lambda = F_{q,t}(D_\lambda)$ for some bigraded S_n-module D_λ

How? By [Haiman, Bridgeland-King-Reid]

as a stack

$$D^b Coh_T(Hilb_n(\mathbb{C}^2)) \simeq D^b Coh_T([(\mathbb{C}^2)^n / S_n])$$

T acts by scaling w/ weights q, t

$$\Rightarrow \quad K_T(Hilb(\mathbb{C}^2)) \simeq \bigoplus_n \underbrace{K_{T\times S_n}((\mathbb{C}^2)^n)}_{\substack{\parallel \\ K_{T\times S_n}(pt)}}$$

$$\Rightarrow \quad K_T(Hilb(\mathbb{C}^2)) \otimes_{K_T(pt)} \mathbb{Q}(q,t) \simeq \Lambda_{q,t}$$

we'll find D_λ here.