Equivariant K-theory: Lecture 16

June-16-22
3:45 PM

Nakajima quiver variety of a quiver Q:

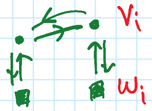
alg. symplectic reduction.

$$Q \rightsquigarrow Q^{\text{doubled}}_{\text{framed}} \rightsquigarrow \mathrm{Rep}\left(Q^{\text{doubled}}_{\text{framed}}\right) \rightsquigarrow \mathcal{M}_Q := \mathrm{Rep} /\!\!/\!\!/_{\theta}\, G_v$$

$$\circlearrowleft \quad G_v := \prod_i GL(V_i)$$

$$\mathcal{M}_Q = \bigsqcup_{\vec{v},\vec{w}} \mathcal{M}_Q(\vec{v},\vec{w})$$

$\uparrow \uparrow$ dimensions of V_i & W_i



E.g. $Q = \bullet$

for generic θ

$$\mathcal{M}_Q(v,w) \cong T^*Gr(v,w)$$

$$U_q\hat{\mathfrak{g}}_Q \cong U_q\hat{\mathfrak{gl}}_2$$

Note: $G_W := \prod_i GL(W_i) \curvearrowright \mathcal{M}_Q$ has isolated fixed points.

$\mathbb{C}^\times_q :=$ scaling the symplectic form

Let $A \subset G_W$ be the max. torus. $T := A \times \mathbb{C}^\times_q$.

$$\Rightarrow \quad K_T(\mathcal{M}_Q)_{loc} \simeq \bigoplus_p K_T(\{p\})_{loc}$$

are very easily-understood reps. of $U_q\hat{\mathfrak{g}}_Q$.

now over $K_T(pt)$ not just $K_{\mathbb{C}^\times_q}(pt) = \mathbb{Z}[q^{\pm}]$

Thm [Nakajima]:

$\{K_T(\mathcal{M}_Q(\vec{w}))\}_{\vec{w}}$ is complete set of f.d. $U_q\hat{\mathfrak{g}}_Q$-mods

$$K_T(\mathcal{M}_Q(\vec{w})) = \bigsqcup_{\vec{v}} K_T(\mathcal{M}_Q(\vec{v},\vec{w}))$$

$\uparrow$ they span in Grothendieck ring.

Recall from last time: $U_q\hat{\mathfrak{g}}_Q \simeq K_T(Z_Q)$

Steinberg $\mathcal{M}_Q \times_{\mathcal{M}_Q^0} \mathcal{M}_Q$

$\uparrow$ Thm of Nakajima for $Q =$ finite ADE type.

Another (more general) view of Nakajima's thm:

Instead of constructing $U_q\hat{\mathfrak{g}}_Q$ "by hand",

find a geometric construction of its R-matrix. !

find a geometric construction of its R-matrix. !

Maulik-Okounkov

(works for arbitrary Q.)

element in $\mathrm{End}(V \otimes W)$ $\forall\ \mathcal{U}_q\hat{\mathfrak{g}}$-mods V, W

as vector spaces

$$\| $$

$$\mathrm{End}\big(K_T(\mathcal{M}_Q(\vec w)) \otimes K_T(\mathcal{M}_Q(\vec w'))\big)$$

Need some facts about $\mathcal{M}_Q$:

1. Let $\begin{pmatrix} \overbrace{1 \ \cdots \ 1}^{\vec w_1} & \\ & \overbrace{a \ \cdots \ a}^{\vec w_2} \end{pmatrix} \in \mathbb{C}^\times_a \subset A$

then $\mathcal{M}_Q(\vec w)^{\mathbb{C}^\times_a} = \mathcal{M}_Q(\vec w_1) \times \mathcal{M}_Q(\vec w_2)$

$$\Rightarrow \quad K_T(\mathcal{M}_Q(\vec w))_{loc} \simeq \bigotimes_i K_T(\mathcal{M}_Q(\underset{\uparrow}{\delta_i}))^{\otimes w_i}_{loc}$$

$(0, \ldots, 0, 1, 0, \ldots, 0)$

call this module

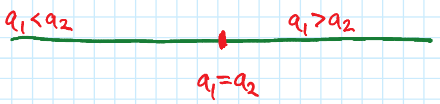
$$F_i(a_i) := K_T(\mathcal{M}_Q(\delta_i))$$

"evaluation representations"

module still depends on the i^{th} equivariant wt in $(a_1, \ldots, a_n) \in A$.

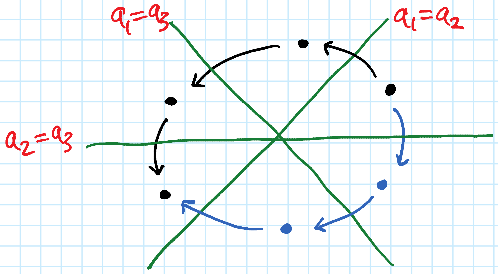
2. Desired R-matrices are maps

$$F(a_1) \otimes F(a_2) \longrightarrow F(a_2) \otimes F(a_1)$$



Lie $A = (\mathbb{C}^\times)^2$

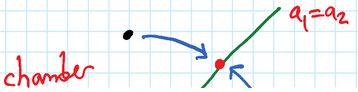
YBE is the equality

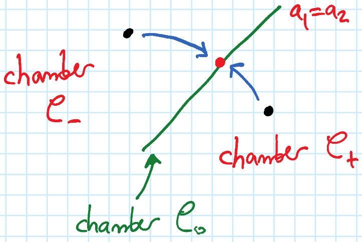


Lie $A = (\mathbb{C}^\times)^3$

The key idea of Maulik-Okounkov:

should exist a factorization of R-matrices as:





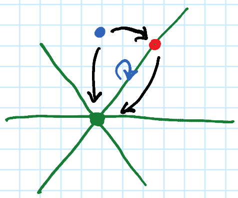
$$R_{\mathfrak{C}_+\to\mathfrak{C}_-} = S^{-1}_{\mathfrak{C}_-\to\mathfrak{C}_0}\; S_{\mathfrak{C}_+\to\mathfrak{C}_0}$$

for maps $S_{\mathfrak{C}\to\mathfrak{C}'} : K_T(X^{a}) \to K_T(X^{a'})$

↗ which should exist for any specialization $\mathfrak{C} \rightsquigarrow \mathfrak{C}'$

generic elements in $\mathfrak{C}, \mathfrak{C}'$ respectively.

To satisfy YBE, require these S satisfy triangle lemma:



⇒ suffices to construct

$a\in\mathfrak{C}$ generic

$$S_{\mathfrak{C}} := S_{\mathfrak{C}\to 0} : K_T(X^{a}) \to K_T(X)$$

for any chamber $\mathfrak{C}$.

What properties should $S_{\mathfrak{C}}$ have?

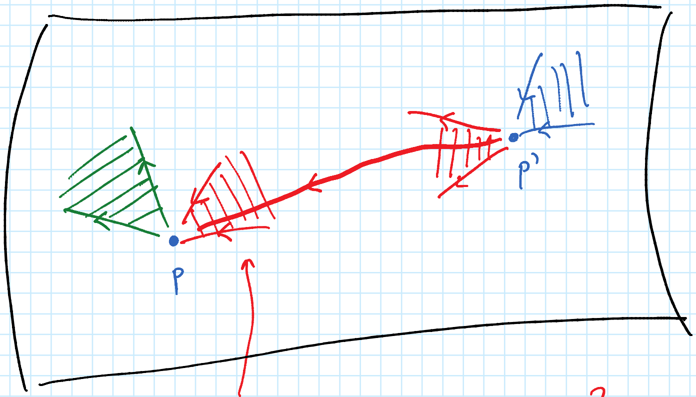
① upper-triangular wrt. some ordering.
(so $R = S_-^{-1} S_+$ is some LU factorization)
$S_{\mathfrak{C}_-}$ and $S_{\mathfrak{C}_+}$

② $S_-\big|_{q=1} = S_+\big|_{q=1}$ so that $R\big|_{q=1} = \mathrm{id}.$ is the trivial braiding.

e.g. $\mathcal{U}_q\mathfrak{g}\big|_{q=1} = \mathcal{U}\mathfrak{g}$. is cocomm.

First insight: attracting manifolds of fixed points.
↑ like in Morse theory.

think about $\mathfrak{C}$ generic for simplicity.



$\{x\in X : \lim_{t\to\infty} \sigma(t)\cdot x = p\}$
$\sigma(t)\in\mathfrak{C}$ cochar.

"repelling"

$\{x\in X : \lim_{t\to 0} \sigma(t)\cdot x = p\}$
$\sigma(t)\in\mathfrak{C}$ cocharacter

"attracting"

$$\Rightarrow \quad \mathrm{Attr}_+ := \{(p,x) : x \text{ attracted to } p\} \subset X^A \times X$$

(similarly have Attr_-)

satisfy $T_p \mathrm{Attr}_+(p) = T_p \mathrm{Attr}_-(p)^* \otimes q$ ← wt of sympl. form.

$\mathrm{Attr}_+|_{p\times X}$

$$\underset{w}{\in} \longleftrightarrow \underset{qw^{-1}}{\in}$$

$$\Rightarrow \begin{cases} \mathcal{O}_{\mathrm{Attr}_+}|_{\Delta(X^A)} = \prod_w \frac{1}{1-w} \\ \mathcal{O}_{\mathrm{Attr}_-}|_{\Delta(X^A)} = \prod_w \frac{1}{1-qw^{-1}} \end{cases}$$

almost satisfies ② on the diagonal $\Delta(X^A)$

Furthermore, $\mathcal{O}_{\mathrm{Attr}_+}$ is upper-triangular in "attracting ordering":

$$\underset{p,p' \in X^A}{p' < p} \iff p' \in \overline{\mathrm{Attr}_+(p)}$$

& $\mathcal{O}_{\mathrm{Attr}_-}$ is lower-triangular.

Corrections necessary:

closed in $X^A \times X$

- replace $\mathrm{Attr} \rightsquigarrow \overline{\mathrm{Attr}} \rightsquigarrow \overline{\mathrm{Attr}}^f$ (better-behaved in families).

- "symmetrize" $\mathcal{O}_{\overline{\mathrm{Attr}}^f} \rightsquigarrow \mathcal{O}_{\overline{\mathrm{Attr}}^f} \otimes (\mathcal{N}^\vee_{\overline{\mathrm{Attr}}^f/X})^{1/2}$

$$\prod_w \frac{1}{1-w} \rightsquigarrow \prod_w \frac{1}{w^{1/2}-w^{-1/2}}$$

(satisfies property ② along the diagonal.)

- do some off-diagonal corrections to make ② hold.

(e.g. all off-diagonal entries in S contain $(1-q)$.)

(??) ← a certain degree bound of $S|_{p\times X}$ in terms $S|_{p'\times X}$ for $p' < p$.

Thm: [Maulik–Okounkov, Okounkov–Smirnov]

Such S_ϵ exist & are uniquely characterized by

1. upper-triangularity & $S_\epsilon|_{\Delta(X^A)} \propto \mathcal{O}_{\mathrm{Attr}_\epsilon}$

"stable envelopes".

"stable envelopes":

1. upper-triangularity & $S_e|_{\Delta(X^A)} \propto \mathcal{O}_{\overline{\mathrm{Attr}}_e}$
2. the degree bound