Last time:

$$f_*[\mathcal{E}] = \sum_k (-1)^k [R^k f_* \mathcal{E}]$$
$$f^*[\mathcal{F}] = \sum_k (-1)^k [L^k f^* \mathcal{F}]$$

need finiteness

Pushforward: $f: X \xhookrightarrow{\iota} X \times Y \xrightarrow{\pi} Y$

ι closed embedding $\Rightarrow$ $R^{>0}\iota_* \mathcal{F} = 0$

regular " — $\Rightarrow$ $\underset{\substack{\cap \\ K_G^{vect}(Y)}}{\iota_* \mathcal{F}}$ has Koszul resolution.

Prototypical example: $\iota: X \hookrightarrow E$ zero section of a vector bundle.

$$\cdots \to \pi^* \Lambda^2 E^\vee \to \pi^* E^\vee \to \mathcal{O}_E \to \iota_* \mathcal{O}_X \to 0$$

check exactness @ stalks $\equiv$ resolution of $\iota_* \mathcal{O}_0$ in $\mathbb{A}^n$

π is projection: e.g. $\pi: X \to pt$ has:

$$\pi_*[\mathcal{F}] = \underbrace{\sum_k (-1)^k [H^k(X, \mathcal{F})]}_{\chi(X, \mathcal{F})}$$

K-theoretic analogue of $\omega \mapsto \int_X \omega$

Euler characteristic.

More generally, if $\pi: X \times Y \to Y$,

π_* is $\chi(X, -)$ by base change theorem.

$$\chi(X, \mathcal{F}|_y) = (\pi_*[\mathcal{F}])_y \in K_G(\{y\})$$

Pullback: $f: X \xhookrightarrow{\iota} X \times Y \xrightarrow{\pi} Y$

1. π is flat $\Rightarrow$ π^* is exact $\Rightarrow$ $\pi^*[\mathcal{F}] = [\pi^* \mathcal{F}]$

(or any f, but $\mathcal{F}$ is a vector bundle)

2. ι closed embedding: $\iota: X \hookrightarrow Y$

$$\iota^* \mathcal{F} := \mathcal{O}_X \otimes_{\iota^{-1}\mathcal{O}_Y} \iota^{-1}\mathcal{F} = \iota_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{F}$$

$$\iota^*[\mathcal{F}] = \sum (-1)^k [\mathcal{T}or_k^{Y}(\iota_* \mathcal{O}_X, \mathcal{F})]$$

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$$\iota^*[\mathcal{F}] = \sum_k (-1)^k \left[\mathcal{T}or_k^Y\left(\iota_* \mathcal{O}_X, \mathcal{F}\right)\right]$$

a. if Y is smooth $\Rightarrow$ $\mathcal{F}$ has finite resolution by vect. bunds.

$\left(K_G^{\mathrm{vect}}(Y) = K_G(Y) \right)$

b. ι regular $\Rightarrow$ $\iota_* \mathcal{O}_X$ has a Koszul resolution.

$\Rightarrow$ In general, $K_G(X)$ has $\begin{cases} \text{proper pushforward} \\ \text{flat pullback} \end{cases}$

((like Chow, more generally "Borel-Moore"-type homology theory)

$K_G^{\mathrm{vect}}(X)$ has arbitrary pullback

Tensor product : $\boxtimes : K_G(X) \otimes K_G(Y) \to K_G(X \times Y)$ always exists.

$- \otimes - = \Delta^* (- \boxtimes -)$ exists whenever Δ^* does.

$\Delta : X \hookrightarrow X \times X$.

for $K_G(X)$; can always do $\otimes : K_G^{\mathrm{vect}}(X) \otimes K_G(X) \to K_G(X)$

$\Rightarrow$ $K_G(X)$ is a $K_G^{\mathrm{vect}}(X)$-module

$K_G^{\mathrm{vect}}(X)$ is also a ring.

$\Rightarrow$ $K_G(X)$, $K_G^{\mathrm{vect}}(X)$ are all $K_G(pt)$-modules

$K_G(pt) = R(G)$

$K_G(pt) \xrightarrow{\pi^*} K_G^{\mathrm{vect}}(X)$

trivial vector bundles on X with non-trivial equivariant wt.

Push-pull : $X \times Y \supset Z$ a "correspondence" between X & Y

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

an example

e.g. $Gr(k,n) \times \mathbb{P}^{n-1} \supset \{ (V, L) : L \in V \}$

$(k-1)$-dim linear subspaces in $\mathbb{P}^{n-1}$

$$\begin{tikzcd} & \{(V,L)\} \arrow[dl] \arrow[dr] & \\ Gr(k,n) & & \mathbb{P}^{n-1} \end{tikzcd}$$

(k-1)-dim linear subspaces in $\mathbb{P}^{n-1}$

$Gr(k,n)$ $\qquad\qquad$ $\mathbb{P}^{n-1}$

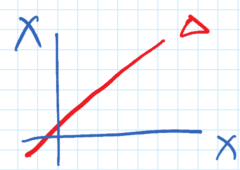
A source of operators $K_G(Y) \to K_G(X)$

$$\pi_{1*}\Big(\underbrace{\mathcal{E}}_{\text{called a kernel}} \otimes \pi_2^*(-)\Big)$$

some sort of convolution

$\mathcal{E} \in K_G(X \times Y)$ is the K-theoretic replacement for $Z \subset X \times Y$

in $D^b Coh$, this is known as a Fourier–Mukai transform.



E.g. $\Delta \subset X \times X$ diagonal

$$\pi_{1*}\left(\mathcal{O}_\Delta \otimes \pi_2^*(-)\right) \overset{\text{(exercise)}}{=} id \circlearrowright K_G(X)$$

very powerful already: if $\exists$ a finite resolution

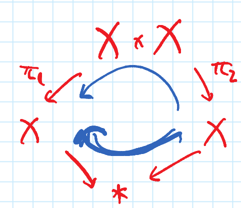
$$\cdots \to \underbrace{\mathcal{F}_1 \boxtimes \mathcal{G}_1} \to \underbrace{\mathcal{F}_0 \boxtimes \mathcal{G}_0} \to \mathcal{O}_\Delta \to 0 \qquad \text{on } X \times X$$

then:

$$K_G(X) \ni \mathcal{F} = id \cdot \mathcal{F} = \sum_i (-1)^i \pi_{1*}\left(\pi_1^*\mathcal{F}_i \otimes \pi_2^*\mathcal{G}_i \otimes \pi_2^*(\mathcal{F})\right)$$

$$= \sum_i (-1)^i \mathcal{F}_i \otimes \pi_{1*}\pi_2^*(\mathcal{G}_i \otimes \mathcal{F}) \quad \text{projection formula.}$$

$$= \sum_i (-1)^i \mathcal{F}_i \otimes \chi(X, \mathcal{G}_i \otimes \mathcal{F})$$



★★ $\Rightarrow$ $K_G(X)$ is spanned as a $K_G(pt)$-mod by $\{\mathcal{F}_i\}$

Exercise: find a Koszul resolution of $\mathcal{O}_\Delta \to \mathbb{P}^n \times \mathbb{P}^n$
(Beilinson resolution.)

$\Rightarrow$ $K_G(\mathbb{P}^n)$ is spanned by $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-1)$.

More generally, the same is true for $\underline{\mathbb{P}(\mathcal{E})} \to X$ $\quad$ ($K_G(\mathbb{P}(\mathcal{E}))$ as a $K_G(X)$-module.)
projective bundle.

In contrast, e.g. no such resolution exists for $\mathcal{O}_\Delta \to \underline{E} \times E$
elliptic curve.

since $K(E) \supset Pic(E)$

↑ already infinitely gen'd as a $\mathbb{Z}$-module.

$$\left(\Rightarrow \quad K(E \times E) \neq K(E) \otimes K(E) \right)$$

<u>Thom isomorphism thm</u>: $\pi : E \to X$ affine bundle (rk n.)

$$\pi^* : K_G(X) \xrightarrow{\sim} K_G(E) \quad \text{is } \underline{\text{iso}}.$$

<u>Remark</u>: really <u>need</u> K_G here

e.g. no iso for K_G^{vect} or Pic ← (exercise)

$$\Rightarrow \quad K_G(\mathbb{C}^n) \simeq K_G(pt) = R(G).$$