K-theory vs. cohomology

Note: there are two different worlds:

	topological	algebraic	
	$K_G^{Top}(X)$	$K_G(X)$	
	$H_G^*(X)$	$A_G^*(X)$	(Chow)

Our discussion today holds equally well in either world.

e.g. E = elliptic curve

$H^2(E) = \mathbb{Z}$ $\qquad$ $A^1(E) \cong E \oplus \mathbb{Z}$

"homological" equivalence $\Leftarrow$ "rational" equivalence

G stands for both

compact Lie group $\equiv$ (equiv. of cat.) reductive alg grp.

e.g. $U(1)$ $\qquad$ $\mathbb{C}^*$

A non-equivariant comparison:

suppose X smooth variety $/\mathbb{C}$.

Chern character

$$ch : \underbrace{K(X)}_{= K^{Vect}} \longrightarrow A^*(X) \otimes \mathbb{Q} \quad \text{has denominators.}$$

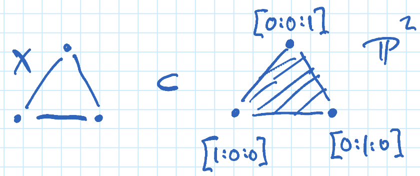
$$\mathcal{O}(D) \longmapsto \exp(D) = \sum_{k \geq 0} D^k / k!$$

$$\left.\begin{array}{l} \mathcal{L}_1 \oplus \mathcal{L}_2 \longmapsto ch(\mathcal{L}_1) + ch(\mathcal{L}_2) \\ \mathcal{L}_1 \otimes \mathcal{L}_2 \longmapsto ch(\mathcal{L}_1) \cdot ch(\mathcal{L}_2) \end{array}\right\} \text{ ring hom.}$$

(intersection product.)

Why the exponential?

e.g. $X = \{xyz = 0\} \subset \mathbb{P}^2$



$$[X] = 3 \cdot [\text{line}] \in A^*$$

$$[\mathcal{O}_X] = 3 \cdot [\mathcal{O}_{\text{line}}] - 3 \cdot [\mathcal{O}_{pt}] \in K$$

$\Rightarrow$ $[\mathcal{O}_{\text{line}}] \mapsto [\text{line}]$ loses intersection information.

Thm: For smooth varieties $/\mathbb{C}$: $ch \otimes \mathbb{Q} : K(X) \otimes \mathbb{Q} \xrightarrow{\sim} A^*(X) \otimes \mathbb{Q}$

is an iso of $\mathbb{Q}$-algs

with $ch(F_i K) \subset A_{\leq i} \otimes \mathbb{Q}$

filtration by dim supp $\qquad$ filtered by dim.

$F_i K(X) := \{ \mathcal{F} : \dim \operatorname{supp} \mathcal{F} \leq i \}$

$$A_* = A^{\dim X - *}$$

(More general versions, for arbitrary schemes, in [Fulton].)

Note: strong finiteness properties on both sides

$$\mathcal{O}(D) \xmapsto{\ ch\ } \exp(D) = \sum_{k\geq 0} D^k/k! \qquad \text{(} = 0 \text{ for } k \gg 0 \text{ by dimension.)}$$

$$-\sum_{k\geq 0} (1-\mathcal{O}(D))^k/k = \log \mathcal{O}(D) \longmapsfrom D \qquad \text{(} = 0 \text{ for } k \gg 0 \text{ by dim supp)}$$

Pf. idea: Show that $ch = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$ w.r.t. filtrations.

1. $F_i K / F_{i-1} K$ is spanned by $[\iota_* \mathcal{O}_Z]$, $\iota : Z \hookrightarrow X$ dim i subvariety (exercise, use excision)

2. If ι regular,

   Todd class: cohom analogue of $\Lambda_{-1}^{\bullet}(N_{X/Z}^{\vee})$

   explicitly, $\prod_x \frac{x}{1-e^{-x}}$ ← cohom. Chern roots.

   $$ch(\iota_* \mathcal{O}_Z) = \iota_* ch_* \left(td(N_{X/Z})^{-1} \cdot [Z] \right)$$
   $$= \iota_* [Z] + (\text{lower dim. terms})$$

   a form of Grothendieck-Riemann-Roch.

3. If ι not regular, pick $Y \subset Z$ closed s.t. $Z \setminus Y \subset X \setminus Y$ is regular. and use excision. ← also exists in A^* □.

Slogan: $K(X)$ is a cohom. theory associated to $\mathbb{C}^\times$

$A^*(X)$ " ―――――――――――― $\mathbb{C}$

$\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$ exp is an isomorphism.

(More on this next week.)

Q: why work in K at all? why not just A^* or H^*?

A: equivariant methods are very powerful.

An equivariant comparison:

Simplest case: $T = \mathbb{C}^\times$ ⇝ $K_T(pt) = \mathbb{Z}[t^{\pm}]$ "multiplicative" weight

⇝ $A^*_T(pt) = \mathbb{Z}[x]$ (fact.) "additive" weight

$$t \longmapsto e^x = \sum_{k\geq 0} x^k/k! \quad \leftarrow \text{no longer terminating!}$$

also not terminating

also not terminating

$$t \longmapsto e^x = \sum_{k\geq 0} x^k/k!$$ ← no longer terminating!

$$-\sum_{k\geq 0} (1-t)^k/k = \log t \longleftarrow x$$ ← exercise.

Indeed, $\mathbb{Q}[t^{\pm}] \not\cong \mathbb{Q}[x]$ as rings

Another simple case: G = finite group.

$$K_G(pt) \otimes \mathbb{Q} = R(G) \otimes \mathbb{Q} = \text{some nontrivial thing}$$

e.g. $R(\mathbb{Z}/n) \otimes \mathbb{Q} = \mathbb{Q}[t^{\pm}]/(t^n - 1)$

$$H^*_G(pt) \otimes \mathbb{Q} \underset{\text{(fact)}}{=} H^*_{grp}(G;\mathbb{Z}) \otimes \mathbb{Q} = \mathbb{Q} \quad \text{trivial!}$$

surely true for Chow too.

$|G|$ - torsion.

The solution: complete both sides.

[Edidin-Graham]

$$K_G(X) \rightsquigarrow K_G(X)^{\wedge}_I \qquad I := \langle \dim V - V : V \in R(G) \rangle \subset R(G).$$

(augmentation ideal)

$$\bigoplus_i A^i_G(X) = A^*_G(X) \rightsquigarrow \prod_i A^i_G(X)$$

<u>Thm</u>: $\exists$ ch: $(K_G(X) \otimes \mathbb{Q})^{\wedge}_I \xrightarrow{\sim} \prod_i A^i_G(X) \otimes \mathbb{Q}$

e.g. $T = \mathbb{C}^*$, $I = \langle 1-t \rangle \subset \mathbb{Z}[t^{\pm}]$

$$K_T(pt) \otimes \mathbb{Q} = \mathbb{Q}[t^{\pm}] \qquad \begin{aligned} t &= 1-(1-t) \\ t^{-1} &= (1-(1-t))^{-1} = \sum_{k\geq 0} (1-t)^k \end{aligned}$$

$$\rightsquigarrow (K_T(pt) \otimes \mathbb{Q})^{\wedge}_I = \mathbb{Q}[[1-t]]$$

isomorphic with $t = e^x$

and $\prod_i A^i_T(pt) \otimes \mathbb{Q} = \mathbb{Q}[[x]]$

<u>Note</u>: $K_T(pt)^{\wedge}_I \otimes \mathbb{Q} = \mathbb{Q} \otimes \mathbb{Z}[[1-t]] \subsetneq \mathbb{Q}[[1-t]] \ni \sum_{k\geq 0} (1-t)^k/k$

finitely many denominators.

<u>Geometric meaning</u> of $K_G(X)^{\wedge}_I$:

Recall: what is $H^*_G(X)$ (and A^*_G)?

$$\begin{array}{ll} EG & \text{universal } G\text{-bundle.} \\ & (\text{contractible, } G \curvearrowright \text{properly freely}) \\ \downarrow & \\ BG & \text{classifying sp. of } G \end{array}$$

e.g. $BU(1) = \mathbb{C}P^\infty$

$\varinjlim_n \mathbb{C}P^n$

$$H^*_G(X) := H^*(EG \times_G X)$$

"Borel" construction.

[Totaro]

(For A^*_G, replace EG by a limit $\{U : U \subset V\}$)

open sets $G \curvearrowright$ freely

rep. of G

Can do the same construction for K:

Def: $K^{Bor}_G(X) := K^{Top}(EG \times_G X)$

$\exists$ a natural map $K^{Top}_G(X) \xrightarrow{\text{pullback}} K^{Top}_G(EG \times X) = K^{Bor}_G(X)$ (☆)

G freely

Thm: [Atiyah-Segal, Merkurjev] The map (☆) is the completion

for alg. K.

$$K^{Top}_G(X) \to K^{Top}_G(X)^\wedge_I$$