Riemann-Roch for Deligne-Mumford stacks

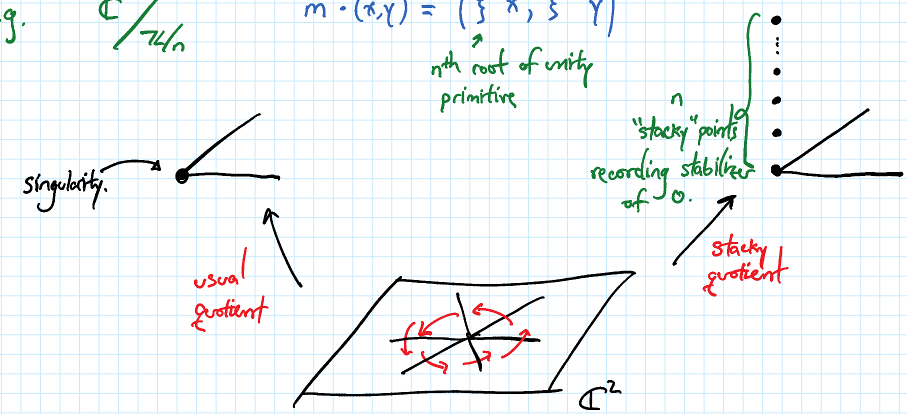
the problem of comparing K-theoretic integral $\chi(X,\mathcal{F})$ to the cohom. integral $\int_X ch(\mathcal{F})\, td(X)$ (X proper smooth variety $/\mathbb{C}$)

What is a stack?

e.g. $\mathbb{C}^2/_{\mathbb{Z}/n}$ $\quad m\cdot(x,y) = (\zeta^m x, \zeta^{-m} y)$

ζ: nth root of unity primitive

as alg. stack. $[\mathbb{C}^2/_{\mathbb{Z}/n}]$



$\mathbb{C}^2/_{\mathbb{Z}/n}$ at 0 : $\operatorname{Spec}\mathbb{C} = \{pt\}$ $\qquad [\mathbb{C}^2/_{\mathbb{Z}/n}]$ at 0 : $[\operatorname{Spec}\mathbb{C}/_{\mathbb{Z}/n}]$

Points of schemes : points $\operatorname{Spec} k$

stacks : a group of "isomorphic" points $[\operatorname{Spec} k/_G]$

In the same way that $G \to *$ is a principal G-bundle, so is $* \to [*/_G]$

e.g. since $K_G(G) = K(pt)$,

$G \curvearrowright G$ by multiplication.

$K_G(*) = K([*/_G])$. $\leftarrow$ an instance of $K([X/_G]) = K_G(X)$.

"Def" : A stack is Deligne-Mumford $\approx$ all stabilizers are finite groups

$\approx$: some technical details hiding here.

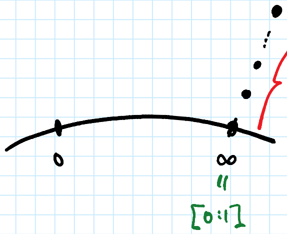
(Generic stabilizer trivial $\Rightarrow$ an "orbifold".)

Our main example : $\mathbb{C}^* \curvearrowright \mathbb{C}^n$ with $\lambda\cdot(x_1,\dots,x_n) = (\lambda^{a_1}x_1,\dots,\lambda^{a_n}x_n)$

$\Rightarrow [\mathbb{C}^n\setminus\{0\}/_{\mathbb{C}^*}] =: \mathbb{P}(a_1,a_2,\dots,a_n)$ "weighted projective space"

$A^* = H^*$

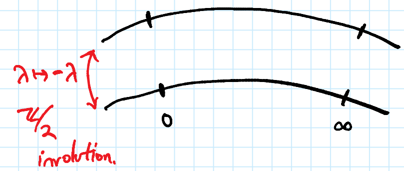
e.g. $\mathbb{P}(1,n)$



$\lambda \cdot (0,1) = (0, \lambda^n \cdot 1) = (0,1)$

λ an n^{th} root of unity.

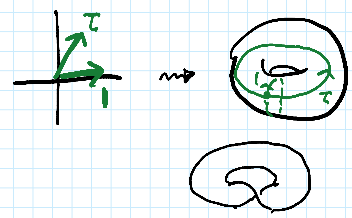
e.g. $\mathbb{P}(2,2)$ (has generic stabilizer)



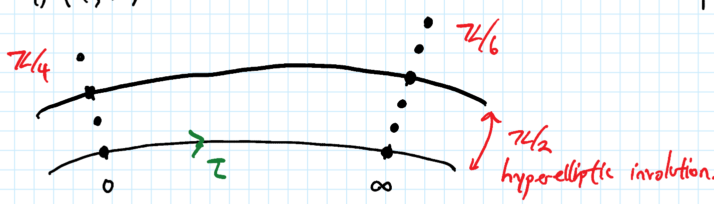
e.g. moduli of elliptic curves $\{ y^2 = x^3 + ax + b \} \subset \overline{\mathcal{M}}_{1,1}$ ← include singular nodal one

↑ each has a hyperelliptic involution $y \mapsto -y$

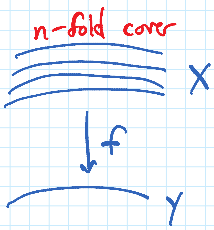
↑ some have extra automorphism (exercise).



Fact: $\overline{\mathcal{M}}_{1,1} = \mathbb{P}(4,6)$ and looks like



Cohom. & K-theory:



$$\int_Y \alpha = \frac{1}{n} \int_X f^* \alpha \quad \Rightarrow \quad \int_{[X/G]} \alpha = \frac{1}{|G|} \int_X \alpha$$

came from X

finite.

aside from these factors, cohom. integrals are insensitive to stackiness

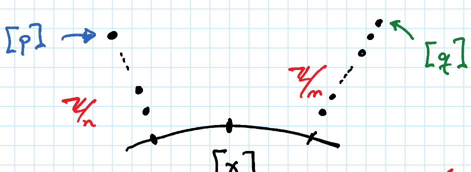
Fact: $H^*(\mathfrak{X}, \mathbb{Q}) = H^*(X, \mathbb{Q})$

DM stack

its "coarse moduli" space ≈ forgot all stabilizers.

e.g. $H^*(\mathbb{P}(n,m), \mathbb{Q})$ has:

assume coprime



$$[p] = \frac{1}{n}[x] \qquad [q] = \frac{1}{m}[x]$$

$$\rightsquigarrow H^*(\mathbb{P}(n,m), \mathbb{Q}) = \mathbb{Q}[x]/x^2 = H^*(\mathbb{P}^1, \mathbb{Q})$$

(All this is compatible with equivariance.)

(Related to how $H^*_G(pt) \otimes \mathbb{Q} = \mathbb{Q}$ for G finite.)

In contrast, e.g.

$T = (\mathbb{C}^\times)^n \qquad K_T(\mathbb{P}(a_1, \dots, a_n)) = \underbrace{\mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}]}_{K_T(pt)}[s^{\pm}] \big/$

previously $a_1 = \dots = a_n = 1$.

$T = (\mathbb{C}^*)^n \curvearrowright$ by usual

$$K_T(\mathbb{P}(a_1,\dots,a_n)) = \mathbb{Z}[t_1^{\pm},\dots,t_n^{\pm}][s^{\pm}] \Big/ \left(\prod_i (1 - s^{a_i} t_i) = 0\right)$$

exact same excision computation as for $\mathbb{P}^n$.

previously $a_1 = \dots = a_n = 1$.

still have Pic generated by $\mathcal{O}(1)$

and $K([*/G]) \xrightarrow{\pi_*} K(*)$ takes G-invariants.

e.g. $K([*/\mathbb{Z}/n]) \longrightarrow K(*)$ is given by $t^a \mapsto \begin{cases} 1 & a = 0 \\ 0 & \text{else} \end{cases}$ $\quad 0 \le a < n$

$$K_{\mathbb{Z}/n}(*) = R(\mathbb{Z}/n) = \mathbb{Z}[t^{\pm}]/(t^n - 1)$$

equivalently $t^a \mapsto \frac{1}{n} \sum_{i=0}^{n-1} (\zeta^i t)^a$

some averaging over G $\quad$ nth root of unity.

χ(DM stacks, $-$) in an example:

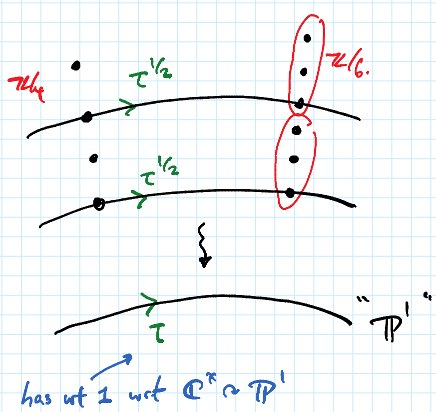
$$\chi\left(\overline{\mathcal{M}}_{1,1}, \frac{1}{1 - Q \cdot \mathcal{O}(1)}\right) \overset{\text{localization (exercise)}}{=} \chi\left([*/\mathbb{Z}/4], \frac{\frac{1}{1 - Q t^{1/4}}}{1 - t^{-1/2}}\right) + \chi\left([*/\mathbb{Z}/6], \frac{\frac{1}{1 - Q (t^{-1})^{-1/6}}}{1 - t^{1/3}}\right)$$

$\overline{\mathcal{M}}_{1,1}$: $\mathbb{P}(4,6)$

$\frac{1}{1 - Q\cdot\mathcal{O}(1)}$: $\sum_{k \ge 0} Q^k \mathcal{O}(k)$

Q: formal variable

$\mathcal{O}(1)$: wt 1 rep on prequotient



$$= \frac{1}{(1 - tQ^4)(1 - tQ^6)}$$

This is the Hilbert series (after $t = 1$) for the ring of modular forms.

$\Rightarrow$ ring $= \mathbb{C}[E_4, E_6]$ (Eigenstein series).

E_4: deg 4, E_6: deg 6

Some general theory:

1. $$\chi([*/\mathbb{Z}/n], f(t)) = \frac{1}{n}\Big(\underbrace{f(t)}_{= \chi(*, f(t))} + \underbrace{f(\zeta t) + f(\zeta^2 t) + \dots + f(\zeta^{n-1} t)}_{\text{"corrections" from non-trivial stacky points}}\Big) \quad (\star)$$

$f(t)$: wt 1 rep.

eigenvalues of $\mathbb{Z}/n \curvearrowright f(t)$.

2. Fact: for G finite,

$$\left(K([X/G]) \otimes \mathbb{Q}\right)^{\wedge}_{-} = \left(K([X/G]) \otimes \mathbb{Q}\right)_{-}$$ localization.

$$\left(K([X/G]) \otimes \mathbb{Q} \right)^{\wedge}_{I} = \left(K([X/G]) \otimes \mathbb{Q} \right)_{I} \quad \text{localization.}$$

$1 - \mathcal{L}$

only finitely many $\mathcal{L}$ involved.

saw last time that this is $\prod_i A^i([X/G]) \otimes \mathbb{Q}$

$= A^*([X/G]) \otimes \mathbb{Q}$ by discussion earlier today.

(in particular $A^{>\dim X} = 0$)

i.e. naive Riemann-Roch only sees non-stacky denominators.

in I → $\dfrac{1}{1-t}$ vs. $\dfrac{1}{1-\zeta t}$ ← not in I.

Def: Inertia stack of $\mathcal{X}$ is

$$I\mathcal{X} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$$

parameterizes (x, ϕ)
$x \in \mathcal{X}$
$\phi \in$ automorphism of x.

e.g. if $\mathcal{X} = [X/G]$

$$I\mathcal{X} = \coprod_{g \in \mathrm{Conj}(G)} [X^g / C(g)]$$

conjugacy classes, centralizer.

$$= \mathcal{X} \sqcup \coprod_{1 \neq g \in \mathrm{Conj}(G)} [X^g / C(g)]$$

"untwisted" sector, "twisted" sector.

For a vector bundle $\mathcal{E}$ on $[X/G]$,

$$\mathrm{tr}\, \mathcal{E} = \sum_\lambda \lambda\, \mathcal{E}_\lambda$$

G-eigenvalue, G-eigenbundle

This is the "coordinate-free" way to write the invariants computation (★)

Thm: [Kawasaki-Riemann-Roch] For $\mathcal{X} = [X/G]$, (DM)

$$\chi(\mathcal{X}, \mathcal{F}) = \int_{I\mathcal{X}} ch\left(\mathrm{tr} \frac{f^* \mathcal{F}}{\lambda_{-1}(N_f^\vee)} \right) td(\mathcal{X})$$

$f: I\mathcal{X} \to \mathcal{X}$ inertia map.

Pf: Apply ordinary RR to each sector of $I\mathcal{X}$. (See [Edidin] for details.)

(Good exercise to compute $\chi(\mathbb{P}(1,n), \mathcal{O}(m))$ via localization
via KRR.
& check that they match.)