Equivariant K-theory: Lecture 8

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Last time:

$$\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \xrightarrow[q = e^{2\pi i \tau}]{/q^{\mathbb{Z}}} E$$

$$H^*(-) \qquad K(-) \qquad \mathrm{Ell}(-)$$

Today: equivariant versions for reductive G.
more concrete.

1. $h^*(X)$ is a module for $h^*(pt)$. $\Rightarrow$ $h_G^*(X) \circlearrowleft h_G^*(pt)$.
any generalized, ... cohom. theory.

For $G = GL(n)$ (for simplicity),

$$H_G^*(pt) = \underset{\text{max. torus}}{H_T^*(pt)^{S_n}} = \mathbb{Z}[x_1, \dots, x_n]^{S_n} = \mathcal{O}(\mathbb{G}_a^n / W)$$

additive group; Weyl group

$$K_G(pt) = K_T^*(pt)^{S_n} = \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}]^{S_n} = \mathcal{O}(\mathbb{G}_m^n / W)$$

$\Rightarrow$ instead of treating cohom as $(\text{space})^{op} \longrightarrow (\text{rings})$ e.g. $H^*(X)$

think instead of $(\text{spaces}) \longrightarrow (\text{schemes})$ (super) e.g. Spec $H^*(X)$

$\mathcal{O}(E^n/W)$ proper, so no global functions.

$$\begin{array}{ccc}
\operatorname{Spec} H_G^*(X) & \operatorname{Spec} K_G(X) & \mathrm{Ell}_G(X) \\
\downarrow & \downarrow & \downarrow \\
\operatorname{Spec} H_G^*(pt) & \operatorname{Spec} K_G(pt) & \mathrm{Ell}_G(pt) \\
\| & \| & \| \\
\mathbb{G}_a^n/W & \mathbb{G}_m^n/W & E^n/W
\end{array}$$

not affine, no underlying ring anymore.

Note: $H_G^*(X)$ and $K_G(X)$ were contravariant wrt. $X_1 \to X_2$ with $G_1 \circlearrowright X_1$, $G_2 \circlearrowright X_2$

$\mathrm{Ell}_G(X)$ is covariant, e.g.

$$1 \to \underset{3^{rd}\text{ roots of unity}}{\mu_3} \to GL(1) \xrightarrow{(-)^3} GL(1) \to 1 \quad \Big\} \text{ apply } \mathrm{Ell}_{(-)}(pt)$$

$$\Rightarrow \quad 0 \to E[3] \to E \xrightarrow{\cdot 3} E \to 0$$

$\uparrow$ 3-torsion on E.

2. An element in K_G $\equiv$ a regular function on Spec K_G.

" ________ Ell_G $:=$ a section of a line bundle on Ell_G

$\uparrow$ there are no nontrivial global functions.

e.g. $Ell_{GL(1)}(pt) = E = \mathbb{C}^\times / q^{\mathbb{Z}}$ $\longleftarrow$ "Tate elliptic curve"

"factor of automorphy"

$s \in H^0(E, \mathcal{L}) \equiv$ a function $s(z)$ satisfying $\underbrace{s(qz) = f(q,z)\, s(z)}$.

e.g. $\vartheta(z) := (z^{1/2} - z^{-1/2}) \prod_{k>0} (1 - q^k z)(1 - q^k/z)$ "q-difference equations"

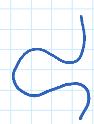
is the odd Jacobi theta, $\vartheta(qz) = -q^{-1/2} z^{-1} \vartheta(z)$

$\Rightarrow$ it is a section of $\mathcal{O}([0])$

Note: $\lim_{q \to 0} \vartheta(z) = z^{1/2} - z^{-1/2}$ = a symmetrized version of $1 - z^{-1} = \Lambda_{-1}(z^{-1})$

elliptic $\longleftarrow$ not a coincidence $\longrightarrow$ K-theoretic

There are degenerations

$\{y^2 = x^3 + ax + b\}$ 

 $= \{y^2 = x^3 + x\}$ nodal cubic

 $= \{y^2 = x^3\}$ cuspidal cubic

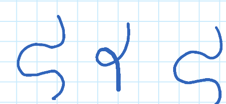
Recall/fact: $\mathrm{Pic}^0(E^\vee) = E$

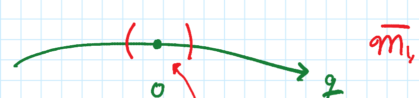
$\mathrm{Pic}^0(\text{nodal}) = \mathbb{G}_m$

$\mathrm{Pic}^0(\text{cusp}) = \mathbb{G}_a$

exactly passage from Ell $\rightsquigarrow$ K, H^*

In particular, $\lim_{q \to 0} \mathbb{C}^\times / q^{\mathbb{Z}}$ = nodal. !





$(G = GL(n))$.

3. What does $Ell_G(X) \to E^n/W$ look like? K-theory lives here.

Take $G = T$ the maximal torus.

$U \times \mathrm{Spec}\, K(X^{T'}) \to \mathrm{Spec}\, K_{T'}(X) \to \mathrm{Spec}\, K_T(X)$

$$\begin{tikzcd}
U \times \operatorname{Spec} K(X^{T'}) \arrow[r] \arrow[d] & \operatorname{Spec} K_{T'}(X) \arrow[r] \arrow[d] & \operatorname{Spec} K_T(X) \arrow[d] \\
U \arrow[r, hook, "\text{localization}"'] & T' \arrow[r, hook, "\text{covariance wrt. } T' \subset T"'] & T
\end{tikzcd}$$

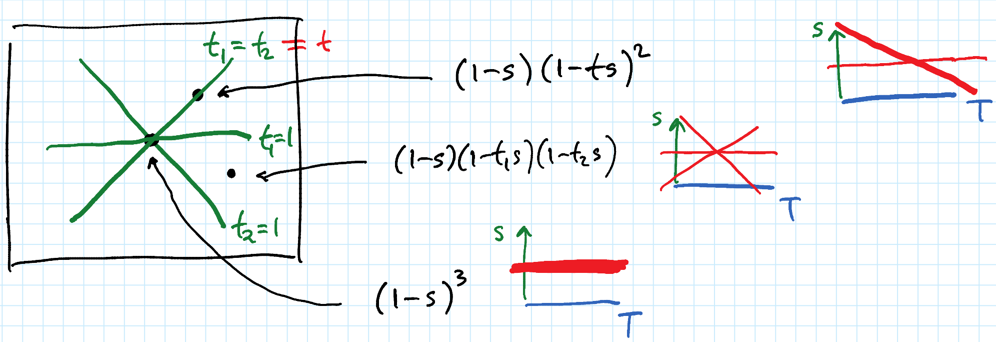
all $t \in T'$ generate a dense subgrp of T' → U

$$\Rightarrow \quad \begin{tikzcd}
\operatorname{Spec} K(X^t) \arrow[r] \arrow[d] & \operatorname{Spec} K_T(X) \arrow[d] \\
\{t\} \arrow[r, hook] & T
\end{tikzcd}$$

Note: generically, $X^t = X^T$ but for special t it may be larger.

$\begin{pmatrix} 1 & & \\ & t_1 & \\ & & t_2 \end{pmatrix} \in GL(3)$.

e.g. $K_T(\mathbb{P}^2) = \mathbb{Z}[t_1^{\pm}, t_2^{\pm}][s^{\pm}] \Big/ (1-s)(1-t_1 s)(1-t_2 s)$.

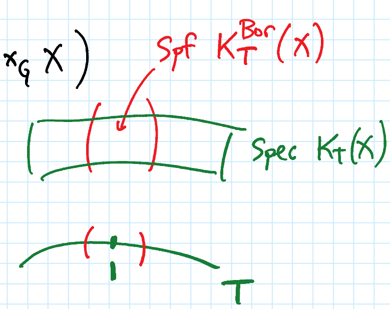


$\operatorname{Spec} K_T(pt) = T$

$\Rightarrow$ $\operatorname{Spec} K_T(\mathbb{P}^2) = 3$ copies of T glued in specific ways along a wall-and-chamber arrangement in the base T.

exact same is true for $\operatorname{Spec} H_T^*$ and Ell_T by replacing $\mathbb{G}_m$ with $\mathbb{G}_a$ and E.

Borel-equivariant constructions, e.g. $K_G^{Bor}(X) = K^{Top}(EG \times_G X)$ live only over a formal neighborhood of $1 \in T$



Remark: only place in practice where working in K_T differs from working in Ell_T is pushforwards.

$\iota : X \hookrightarrow Y$ (normal bundle N) $\rightsquigarrow$ ι_* (K-th.) $\approx \cdot \prod_{w \in N} (1 - w^{-1}) : \mathcal{O}_{\operatorname{Spec} K_T} \to \mathcal{O}_{\operatorname{Spec} K_T}$

$\iota_* \approx \cdot \prod_{w \in N} \vartheta(w) : \Theta(-N) \to \mathcal{O}_{Ell_T}$

bundle N

$$\iota_* \approx \cdot \prod_{w \in N} \underbrace{\vartheta(w)}\ :\ \Theta(-N) \to \mathcal{O}_{Ell_T}$$

elliptic

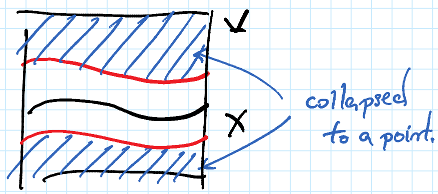
have nontrivial degree
(sections of a nontrivial line bundle on $Ell_T(pt)$)

(& similarly for $\pi: X \to Y$ projection)

This $\Theta(-N)$ is the elliptic Thom sheaf

in H^* & K, was trivial
in Ell, nontrivial.

(Thom isomorphism, see e.g. [Milnor–Stasheff])
$h^*(X) \xrightarrow{\sim} h^*(\mathrm{Thom}(V))$ $\forall$ v.b. $V \to X$

$\mathrm{Thom}(V) =$  $\equiv \mathbb{P}(\mathcal{O} \oplus V) / \mathbb{P}(V).$

in alg. geom.