Equivariant enumerative geometry of $Hilb(\mathbb{C}^2)$ ← of points.

X smooth projective surface $\rightsquigarrow$ $\mathcal{M}_H(v)$ moduli of stable sheaves on X.

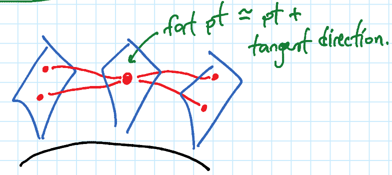
(v ← topological data, e.g. $(rk\,\mathcal{E}, c_1(\mathcal{E}), c_2(\mathcal{E}))$; H ← polarization defining stability.)

A simple case: $v = (1, 0, n)$ $\Rightarrow$ all $[\mathcal{E}] \in \mathcal{M}_H(v)$ are ideal sheaves $\mathcal{I}_Z$ of 0-dim. subscheme $Z \subset X$ of n pts.

$\Rightarrow$ $\mathcal{M}_H(1,0,n) \simeq Hilb_n(X)$

Hilbert scheme of n points on X.

e.g. $Hilb_1(X) = X$, $Hilb_2(X) = Bl_\Delta(X \times X)$ (Δ ↑ diagonal)



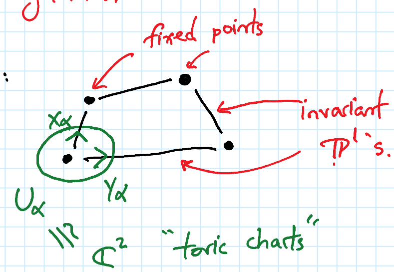
Thm [Fogarty]: $Hilb_n\left(\text{smooth surface}\right)$ is smooth, and

Hilbert-Chow: $HC : Hilb_n(X) \longrightarrow Sym^n X := X^n/S_n$ (remembers only the multiplicities + points.)

is birational. ← a resolution of singularities.

Suppose X is toric. The picture is: ("toric polytope")

$T = (\mathbb{C}^\times)^2$ ↺ X



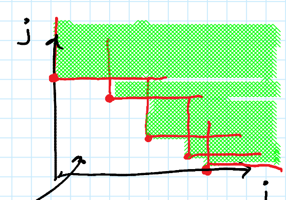
$\Rightarrow$ $T \curvearrowright Hilb(X)$ too!

$Hilb(X) = \bigsqcup_n Hilb_n(X)$

1. Suppose $[\mathcal{I}_Z] \in Hilb_n(X)$ is T-fixed.

$I_\alpha := \mathcal{I}_Z|_{U_\alpha} = \langle f_1, \ldots, f_m \rangle \subset \mathbb{C}[x_\alpha, y_\alpha]$

T-fixed $\Rightarrow$ monomials $x^i y^j$

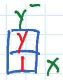


$\Rightarrow$ $\mathcal{O}_Z|_{U_\alpha} = \mathbb{C}[x_\alpha, y_\alpha]/I_\alpha \equiv$ Young diagram of an integer partition.

(↑ dim $n < \infty$)

e.g. $[1 | x]\ x^2 \equiv I = \langle x^2, y \rangle$ is a point in $Hilb_2(\mathbb{C}^2)$.

$[1 / y]\ y^2,\ x \equiv I = \langle x, y^2 \rangle$ is another distinct pt "



$\equiv \quad I = \langle x, y^2 \rangle$ | is another distinct pt "

↖ both map to $2\cdot[0] \in \mathrm{Sym}^2 \mathbb{C}^2$ under HC.

2. Tangent space: $T_Z \, \mathrm{Hilb}(X) = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$ (exercise)

$\mathcal{O}_Z = \mathcal{O}_X / \mathcal{I}_Z$

(like how $T_{[V]} Gr(k, W) = \mathrm{Hom}(V, W/V)$)

Rewrite in terms of $\chi(-,-) := \sum_i (-1)^i \mathrm{Ext}^i(-,-)$ ← easy to compute in $K_T(X)$.

e.g. $\chi(\mathcal{O}_X, -) = \sum_i (-1)^i H^i(-) = \chi(-)$ is usual Euler characteristic.

LES of $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ after applying $\mathrm{Hom}(-, \mathcal{O}_Z)$:

$$0 \to \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \xrightarrow{0} \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \to$$

($\mathcal{I}_Z|_Z = 0$)

$\mathrm{Ext}^1 \qquad \cancel{\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z)}$ ($H^{>\dim Z}(\mathcal{O}_Z) = 0$)

$\mathrm{Ext}^2 \qquad \cancel{\mathrm{Ext}^2(\quad)} \qquad \to 0.$

$$\Rightarrow T_Z = -\chi(\mathcal{O}_Z, \mathcal{O}_Z) + \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) + \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$$

$\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) = \chi(\mathcal{O}_X, \mathcal{O}_Z)$, $\quad \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = \chi(\mathcal{O}_Z, \mathcal{O}_X)$

Serre duality: $\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = \mathrm{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X)^\vee = H^0(\mathcal{O}_Z \otimes K_X)^\vee = \chi(\mathcal{O}_Z \otimes K_X)^\vee$, and $\chi(\mathcal{O}_Z \otimes K_X)^\vee = \chi(\mathcal{O}_Z, \mathcal{O}_X)$

$$= \sum_\alpha \text{contributions from each } U_\alpha$$

$\Rightarrow$ Suffices to compute on each $U_\alpha \simeq \mathbb{C}^2 \circlearrowleft T$ with weights t_1, t_2.

and use $K_T(\mathbb{C}^2)_{loc} \underset{\sim}{\xrightarrow{i^*}} K_T(\{0\})_{loc}$

$$\Rightarrow \chi(\mathcal{F}) = \frac{\mathcal{F}|_0}{(1-t_1^{-1})(1-t_2^{-1})}$$

← $\mathcal{F}$ is a $\mathbb{C}[x,y]$-module ↖ character is $\frac{1}{(1-t_1^{-1})(1-t_2^{-1})}$

$\mathcal{F}|_0$ is a $\mathbb{C}_T$-module.

$$\Rightarrow \chi(\mathcal{F}, \mathcal{G}) = \frac{\mathcal{F}|_0^\vee \otimes \mathcal{G}|_0}{(1-t_1^{-1})(1-t_2^{-1})} = \chi(\mathcal{F})^\vee \chi(\mathcal{G}) \, (1-t_1)(1-t_2).$$

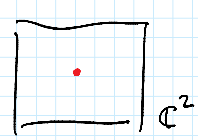
Let $V_\lambda := \chi(\mathcal{O}_{Z_\lambda}) = H^0(\mathcal{O}_{Z_\lambda}) = \sum_{(i,j)\in\lambda} t_1^{-i} t_2^{-j}$

↖ subscheme corresponding to partition λ. ↗

$$T_{Z_\lambda} = V_\lambda + V_\lambda^\vee t_1 t_2 - V_\lambda^\vee V_\lambda (1-t_1)(1-t_2)$$

($t_1 t_2$ = wt of $K_{\mathbb{C}^2}$)

Sanity check: if $\lambda = \square$, $V_\lambda = 1$ and



$$T_{Z_\square} = 1 + t_1 t_2 - (1-t_1)(1-t_2) = t_1 + t_2$$

deg ... $\mathcal{F}|$ ...

$$T_{Z_\square} = 1 + t_1 t_2 - (1-t_1)(1-t_2) = t_1 + t_2$$

Conclusion: $Z_{\mathbb{C}^2}(\mathcal{F}; t_1, t_2, Q) := \chi_{Hilb(\mathbb{C}^2)}(\mathcal{F} \cdot Q^{\deg}) = \sum_\lambda \frac{\mathcal{F}|_{Z_\lambda}}{\Lambda^\bullet_{-1}(T^\vee_{Z_\lambda})} \cdot Q^{|\lambda|} \in K_T(pt)_{loc}[[Q]]$

is a fancy weighted generating function for integer partitions.

non-equivariant specialization $t_1 = t_2 = 1$ is not well-defined ($\exists$ poles there, corresponding to $\mathbb{C}^2$ being noncompact.)

but $Z_X(\mathcal{F}; Q) = \prod Z_{\mathbb{C}^2}(\mathcal{F}|_{U_\alpha}; t_{\alpha_1}, t_{\alpha_2}, Q)$

(same thing as $Z_{\mathbb{C}^2}$ but on X.) (toric charts U_α) (wts of x_α, y_α on U_α)

is OK at $t_1 = t_2 = 1$. ← K-theoretic Donaldson invariant.

$Z_{\mathbb{C}^2}(\mathcal{F})$ is an equivariant "building block" of $Z_X(\mathcal{F})$.

↑ simplest examples of "Nekrasov partition function" for 4d $N=2$ supersymmetric Yang-Mills. (K-theoretic analogues)