MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

PRACTICE FINAL

The exam is **170 minutes**. There are **100 points** in total. No additional material or calculators are allowed.

- Write your **name and UNI** clearly on your exam booklet.
- **Show your work** and reasoning, not just the final answer. Partial credit will be given for correct reasoning, even if the final answer is completely wrong.
- **Don't cheat**!
- Don't panic!

(1) (10 points) State whether the following are true/false. No explanation necessary.
  (a) Some first-order linear systems of ODEs have infinitely many critical points.
  (b) For continuous functions $f(t)$ and $g(t)$,
$$\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$
  (c) Given a square matrix $\mathbf{A}$, there is only one change of basis matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is in Jordan normal form.
  (d) For any square matrices $\mathbf{A}$ and $\mathbf{B}$,
$$\exp(\mathbf{A}+\mathbf{B}) = \exp(\mathbf{A})\exp(\mathbf{B}).$$
  (e) Every linear homogeneous second-order ODE with three regular singular points has a hypergeometric series solution, up to some change of variables.

(2) Consider the first-order equation
$$\frac{dy}{dx} = \frac{x+y+1}{y+1}.$$
  (a) (5 points) Write an autonomous (non-homogeneous) linear 2×2 first-order system associated to this equation, and explain the relationship between the system and the original equation.
  (b) (10 points) Solve the system from part (a).
  (c) (5 points) Draw the slope field of the original equation in a region around $(0,0)$. (Hint: first understand the slope field of the *homogeneous* part of part (a), and then add on the non-homogeneous part.)

(3) Consider the first-order equation
$$y' = \frac{1}{\alpha x + \beta y}.$$
  (a) (5 points) For which α, β is this equation exact? Separable? Linear?
  (b) (5 points) Solve the equation for any non-zero α, β using a change of variables to make it into a separable equation. (Leave your solution in implicit form.)

(4) Consider the second-order equation

$$x^2 y'' - 2e^x y = 0.$$

(a) (7 points) Find all of its singular points (including possibly ∞) and identify whether they are regular or irregular.
(b) (5 points) For one of the singular points in part (a), explain the appropriate series ansatz in order to find a fundamental set of solutions around that point.
(c) (8 points) For one of the singular points in part (a), explain why a series solution must exist and be analytic for all x. Find any series solution around that point up to $O(x^3)$.

(5) Consider the second-order IVP

$$y'' + 4y = \begin{cases} 0 & t < \pi \\ \cos(t) & t \geq \pi \end{cases}, \quad y(0) = y'(0) = 0.$$

(a) (8 points) Solve for y using forward and inverse Laplace transforms.
(b) (7 points) Solve for y using impulse response and convolution.
(c) (5 points) What is the steady-state behavior of y? Can the final value theorem be applied?

(6) Consider the first-order system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, with

$$\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}.$$

(a) (5 points) Find all eigenvectors (and corresponding eigenvalues) of $\mathbf{P}$.
(b) (5 points) Find its Jordan normal form $\mathbf{D}$ and corresponding change of basis matrix $\mathbf{S}$, i.e. so that $\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.
(c) (5 points) For each critical point of the system, describe whether it is stable, asymptotically stable, or unstable. How do solutions behave as $t \to \infty$?
(d) (5 points) Compute the matrix exponential $\exp(\mathbf{D}t)$. Hence write down the general solution to the system of ODEs.

For more practice with Jordan normal form, repeat this problem with the matrix

$$\mathbf{P} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}.$$