MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

HOMEWORK 4 (DUE JUL 22)

Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Explain why, for any constants a, b,

$$(D-a)(D-b)y = (D-b)(D-a)y$$

where $D = d/dx$ is the derivative operator. When $a \neq b$, explain carefully why this means the solutions to

$$(D-a)y_1 = 0, \quad (D-b)y_2 = 0$$

form a fundamental set of solutions for the original equation $(D-a)(D-b)y = 0$.

(2) Find the general solution to

$$y^{(4)} - 2y^{(3)} + 2y' - y = 7e^{2x}.$$

(3) Consider the power series (around $x = 0$)

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

What is its radius of convergence? On the interval where it converges, write $f(x)$ as a rational function. Find the power series around $x = 2$ for this rational function. Explain why this power series is not equal to the original. (This process of "extending" a function beyond its original domain is called **analytic continuation**.)

(4) Compute the power series (around $x = 0$) for $\sin x$ and $\cos x$ and e^x. Explain why they all have infinite radius of convergence. Using this, prove Euler's identity

$$e^{ix} = \cos x + i \sin x$$

for all real numbers x.

(5) For a given constant k, the **Hermite equation** is

$$y'' - 2xy' + 2ky = 0.$$

Find a fundamental set of series solutions. Explain why, when k is a non-negative integer, one of the two series solutions is actually a polynomial. These polynomials are the **Hermite polynomials** $H_k(x)$. What is $H_3(x)$?