MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

HOMEWORK 5 (DUE JUL 25)

Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Around $x = 0$, find the general series solution up to $O(x^6)$ to the equation
$$y' - \cos(x)y = 0.$$

(2) Consider the Cauchy–Euler equation
$$x^2y'' + \alpha xy' + \beta y = 0.$$
Suppose that r is a *repeated* root of indicial equation
$$I(r) := r(r-1) + \alpha r + \beta = 0$$
i.e. $I'(r) = 2r - 1 + \alpha = 0$. Using the ansatz $y = v(x)x^r$, show
$$y_1 = x^r, \quad y_2 = x^r \ln x$$
is a fundamental set of solutions for $x > 0$.

(3) Find the singular points of the **Chebyshev equation**
$$(1 - x^2)y'' - xy' + n^2y = 0.$$
Are they regular or irregular? Pick one of them and explain the series ansatz for the general solution around that point. (Don't actually find the solution.)

(4) The **Bessel equation** (of order one)
$$x^2y'' + xy' + (x^2 - 1)y = 0$$
has a regular singular point at $x = 0$. For the solution $r = 1$ of the indicial equation, show that the series solution around $x = 0$ is
$$y_1 = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!n!2^{2n}}.$$
What happens when you try to find a second solution of the form $y_2 = x^{-1}\sum_{n=0}^{\infty} a_n x^n$ using the other solution $r = -1$ of the indicial equation?

(5) Consider the equation
$$x^3y'' + 2xy' + y = 0.$$
What happens if we try to find a series solution of the form $y = \sum a_n x^n$ around $x = 0$? Why does this happen?