MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

HOMEWORK 6 (DUE JUL 29)

Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) The solutions r_1, r_2 of the indicial equation are called **critical exponents** at the singularity. Find the critical exponents for each regular singular point in
$$(1-x^2)y'' + \sin(x)y' + 3y = 0$$

(2) Find the critical exponents of the hypergeometric equation
$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$$
for the point at infinity. These exponents will depend on a and b. Explain, in all cases, what the correct Frobenius ansatz is in order to obtain a fundamental set of solutions.

(3) Find the singular points of the **Legendre equation** (including possibly at ∞)
$$(1-x^2)y'' - 2xy' + n(n-1)y = 0$$
and determine whether they are regular/irregular. (You can piggyback off your work for the Chebyshev equation in the previous homework.)

(4) The **Legendre functions** $P_n(x)$ are solutions to the Legendre equation. Using problem (3), explain carefully why $P_n(x)$ can be written in terms of the hypergeometric function ${}_2F_1(a, b, c; x)$ for some parameters a, b, c, after a change of variables
$$z = \frac{x-1}{x+1}.$$
(You don't need to find specifically what a, b, c are.)

(5) Recall that the hypergeometric function ${}_2F_1(a, b, c; x)$ is the *analytic* solution to the hypergeometric equation around $x = 0$. Find its radius of convergence in two ways:
   (a) using just the hypergeometric equation and facts about the radius of convergence of series solutions;
   (b) using the ratio test on the explicit formula for ${}_2F_1$.