MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

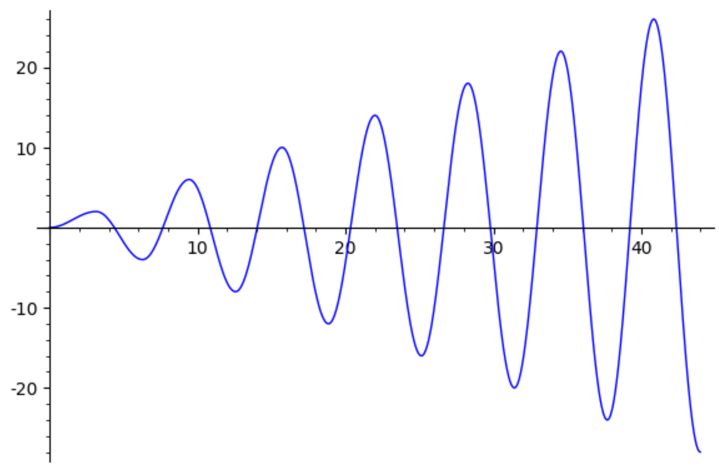
HOMEWORK 8 (DUE AUG 05)

Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Consider the IVP

$$y'' + y = u_0(t) + 2\sum_{k=1}^{N}(-1)^k u_{k\pi}(t), \quad y(0) = 0, \quad y'(0) = 0$$

where N is a large integer. (If you are comfortable with the associated convergence issues, let $N \to \infty$.) Draw a graph of the forcing function. Explain why the solution should look like the following graph, without necessarily solving for y.



(Hint: this phenomenon is called **resonance**.) Then solve the IVP explicitly. Describe which terms in the solution correspond to which parts of the above graph.

(2) Prove that convolution satisfies the following properties:

$$f * g = g * f, \quad f * \delta = f.$$

Here δ is the Dirac delta function.

(3) Using the convolution theorem, find the inverse Laplace transform of

$$Y(s) = \frac{s^2}{(s^2+1)(s^2+4)}.$$

(4) Consider the **Volterra** *integral* equation

$$y(t) + \int_0^t (t-u)y(u)\,du = \sin 2t.$$

Solve for y using the Laplace transform. (Hint: use problem 3 at some point.) Differentiate the equation twice to get the second-order IVP

$$y'' + y = -4\sin 2t, \quad y(0) = 0,\ y'(0) = 2.$$

Show that the expression for $Y(s)$ obtained from this equation is the same.

(5) For the second-order IVP

$$y'' + 4y' + 4y = g(t), \quad y(0) = y'(0) = 0,$$

find its transfer function, its impulse response, and its solution (in terms of g). Assuming $G(s) = \mathcal{L}\{g(t)\}$ is analytic for all s, describe the steady-state behavior of y.