MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

HOMEWORK 9 (DUE AUG 08)

Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

Note: for solutions which are a product of matrices, you do **not** need to multiply everything out. The point of the problem set is not to test how well you can multiply matrices.

(1) Find all linearly independent eigenvectors for the matrix

$$\mathbf{P} := \begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{pmatrix}.$$

(2) Find the Jordan normal form and corresponding basis for the matrix $\mathbf{P}$ in problem 1. (Hint: choose your eigenvectors wisely!) Using this, write the general solution for the homogeneous system $\mathbf{x}' = \mathbf{P}\mathbf{x}$.

(3) Using problem 2, compute the matrix exponential $\exp(\mathbf{P}t)$. Check that its columns agree with the general solution you wrote in problem 2.

(4) The n-th order homogeneous constant-coefficient equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

is equivalent to a system of n first-order equations of the form $\mathbf{x}' = \mathbf{P}\mathbf{x}$. What is the matrix $\mathbf{P}$? (It is called the **companion matrix** of the associated characteristic polynomial; there is a normal form for matrices, called the **rational normal form**, whose blocks are companion matrices instead of Jordan blocks.)

(5) Prove that if $\mathbf{AB} = \mathbf{BA}$, i.e. $\mathbf{A}$ and $\mathbf{B}$ commute with each other, then

$$\mathbf{A}e^{\mathbf{B}} = e^{\mathbf{B}}\mathbf{A},$$

i.e. $\mathbf{A}$ and $e^{\mathbf{B}}$ also commute with each other.