MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

PRACTICE MIDTERM

The exam is **95 minutes**. There are **75 points** in total. No additional material or calculators are allowed.

- Write your **name and UNI** clearly on your exam booklet.
- **Show your work** and reasoning, not just the final answer. Partial credit will be given for correct reasoning, even if the final answer is completely wrong.
- **Don't cheat**!
- Don't panic!

(1) (10 points) State whether the following are true/false. No explanation necessary.
  (a) In the slope field for $y' = f(x, y)$, it is possible to have two different integral curves which intersect each other.
  (b) An n-th order homogeneous linear equation with constant coefficients always has n independent solutions $y_1, \ldots, y_n$ such that the general solution is of the form $c_1 y_1 + \cdots c_n y_n$.
  (c) If y_1, y_2 and $\tilde{y}_1, \tilde{y}_2$ are two fundamental sets of solutions for a homogeneous linear second-order equation, then either

$$\tilde{y}_1 = y_1 \quad \text{and} \quad \tilde{y}_2 = y_2,$$

  or

$$\tilde{y}_1 = y_2 \quad \text{and} \quad \tilde{y}_2 = y_1.$$

  (d) The method of undetermined coefficients says that for $y'' + 2y' + 4y = \cos x$, the appropriate ansatz is $y = A \cos x$ for some unknown constant A.
  (e) If a function is infinitely differentiable, so that its Taylor series exists, then it is an analytic function.

(2) Consider the first-order equation

$$y' = \sin(x) y + x$$

  (a) (5 points) Roughly sketch its slope field in the region $[-5, 5] \times [-5, 5]$. (Hint: this function is odd in x, which saves you half the work.) Draw the integral curve with $y(0) = -1$.
  (b) (5 points) Find the general solution using an integrating factor.

(3) (10 points) Find the value of b so that the equation

$$(e^x \sin y + b e^y \cos x) + (e^x \cos y + 2e^y \sin x) y' = 0$$

is exact. For that value of b, find the general solution.

(4) Consider the non-linear first-order IVP

$$yy' = y^2 + x, \quad y(0) = 2.$$

(a) (5 points) Perform the change of variables $u = y^2 + x$ to simplify the equation. What is the initial condition for u?

(b) (5 points) Using (a), solve the IVP for y by separation of variables.

(5) Consider the second-order equation

$$2y'' - \alpha y' - \alpha^2 y = 0.$$

(a) (5 points) Find the general solution, for all possible α.

(b) (5 points) Show, using the Wronskian, that the solutions you found in (a) actually form a fundamental set of solutions.

(c) (5 points) If the last term were $+\alpha^2 y$ instead of $-\alpha^2 y$, find a fundamental set of *real* solutions.

(6) Consider the second-order equation

$$x^3 y'' + x^2 y' - (e^x - 1)y = 0.$$

(a) (10 points) Find all singular points (including possibly $x = \infty$). For each, determine if it is regular or irregular.

(b) (5 points) Find at least one *analytic* series solution up to $O(x^3)$ at any of the points in (a). Explain where it converges.

(c) (5 points) State an appropriate ansatz for another independent series solution at the point you chose in (b).