All terms \( g \) in \( n \) are packed in a retrieval function.

In invariant counting:

\[ H \psi = \sum g_{\alpha} \psi \]

The parameters \( \psi \) is a formal variable.

In \( g \) invariant:

\[ g_{\alpha} g_{\beta} \]

The parameters \( \psi_{\alpha} = g \mathcal{N} \) is the \( \alpha \) component counting degree.

Perturbative v.s. non-perturbative.

\[ R_{\mathcal{N}} \]

\[ g_{\alpha} = e \mathcal{N} \]

\[ g_{\alpha} = e \mathcal{N} \]

Gut, in counting equation degree.
In CS:

we will explain the expansion w.r.t. \( g_s = \frac{2\pi}{\sqrt{2\pi N}} \)

(the CS perturbation theory)

(In fact \( g_s \))

In quantum group approach to JW inv. (Redeikhin-Turaev),

the quantum group is specialised at \( \sqrt{1} \),

where \( l = 2(\pi + N) \).

(\( l \)th root of 1)

"Nonperturb. ?"

\[ e^{\varepsilon_i} = e^{ig_s} = e^{\frac{2\pi i}{\sqrt{2\pi N}}} = e^{\frac{4\pi i}{2\pi}} = (\sqrt{1})^{\frac{i}{2}} \]
Jones-Witten invariants (or Chern-Simons theory)

Ref: Ohkouchi Quantum invariants, App. F

$M^3$: 3-manifold, cpt, oriented

$G$: compact Lie group e.g. $SU(N)$

simple for simplicity.

$A$: $G$-connection on the trivial bundle $M \times G$

1-form with value in $\mathfrak{g}$

$\mathcal{A}$ = the space of $G$-connections = $\Omega^1(M; \mathfrak{g})$

$CS(A) = \frac{1}{8\pi^2} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$

Chern-Simons functional

(Chern "Tr" x y = (x, y) $\in \mathfrak{g}$

must be understood appropriately).

This is a function (or lagrangian) on $\mathcal{A}$. But

is very different from the usual one in the physics,

as it is independent of the Riemannian metric.

Recall $\frac{dCS}{dA} = 0 \iff A$: flat connection i.e. $F_A = 0$

$\therefore A \iff \text{rep. } \pi_1(M) \to G$. 
$G = \text{the group of bundle automorphism} = \text{Map}(M, G)$ (gauge group in math)

$G \to A$ by gauge transformation

$g^* A = g^{-1} d g + g A g$

$A/G = \text{the space of } G\text{-orbits of } G\text{-connections}
\quad \text{"the space of fields"}

\textbf{NB.} As is usual for a quotient space, it is important to consider also we should consider $A/G$ as a space, as $G$ has stabilizers in general.

Exercise CS is not a function on $A/G$, but

$\text{CS} : A/G \to \mathbb{R}/\mathbb{Z}$ is well-defined.

We now define Jones-Witten invariant
by "quantizing" the Chern-Simons functional:

$\mathfrak{p} \in \mathbb{Z}_{\geq 0} \quad (\text{level})$

$Z_{\mathfrak{p}}(M) \equiv Z_{\mathfrak{p}, G}(M) = \sum_{A/G} DA \exp(\mathfrak{p} i \text{tr CS}(A))$
This is very beautiful formula except that we do not know how to define the path integral.

0 Incorporation of a link

\[ L : \text{link} = \prod_i L_i \text{ (components)} \]

\[ \text{Hol}_{L_i}(A) = \text{the holonomy of } A \text{ along } L_i \]
\[ \in \text{ conjugacy class of } G \]
\[ R_i : \text{finite dimensional representation of } G \]

\[ \Rightarrow \tau_{R_i} \text{ Hol}_{L_i}(A) =: W^{L_i}_{R_i}(A) \]

(Wilson line observable)

\[ Z_{R, G, R_1, \ldots, R_L}(M, L) = \sum_{A \in G} \exp(2\pi i CS(A)) \prod_i \tau_{R_i} W^{L_i}_{R_i}(A) \]

(Jones-Witten link invariant)

This is a “correlation” function in the (quantum) Chern-Simons theory.

Later I will explain the perturbative expansion of the JW invariant, which is relevant for the large \( N \) duality.
But we start with "Hamiltonian approach" to the quantization problem (topological quantum field theory).

We cut $M$ along a surface ($= 2 \dim C^0$-mfld) $\Sigma$, more precisely we should have

$$\forall M_1 = \Sigma \quad \forall M_2 = -\Sigma$$

$$\mathfrak{A}_\Sigma = \text{space of } G\text{-connections on } \Sigma \times G$$

$$\mathfrak{A}_a = \text{space of } G\text{-connections } A \text{ on } M_a,$$

subject to $A|_\Sigma = a$,

$$\sim \text{ker}(g^i \to g_{\Sigma})$$

group of inner automorphisms on $\Sigma \times G$

We expect

$$\mathbb{Z}_\mathbb{R}(M) =$$

$$= \int_{\Sigma} \int_{\mathfrak{a}} \int_{\mathfrak{a}^1} e^{2\pi i \text{CS}(A^1)} \int_{\mathfrak{a}^2} e^{2\pi i \text{CS}(A^2)}$$

$$\frac{\mathfrak{a}}{\mathfrak{g}_{\Sigma}} \quad \frac{\mathfrak{a}^1}{\text{ker}(g^1 \to g_{\Sigma})} \quad \frac{\mathfrak{a}^2}{\text{ker}(g^2 \to g_{\Sigma})}$$
\[ a \to \int_{A^\Sigma} e^{i \int_{g_\Sigma} \text{CS}(A^\Sigma)} \] 

But this is not quite correct.

\[ \Sigma \quad m \quad g \in g_\Sigma \quad \hat{g} \quad \text{its extension to } M' \]

\[ \text{CS} (\hat{g} \ast A^g) - \text{CS}(A^g) = c(a, g) \] (Wess-Zumino term)

Then \[ e^{i c(a, g)} \text{ defines a line bundle } L \]

on \[ \mathcal{A} / g_\Sigma \].

So \[ \ast \text{ a section of the line bundle } L \] on \[ \mathcal{A} / g_\Sigma \].

\[ \mathcal{X}(\Sigma) = \text{the } \textit{Hilbert } \text{ space of such sections} \]

\[ \mathcal{X}(\Sigma) = \mathcal{X}(\Sigma)^* \]

\[ \mathcal{X}(M_1) \in \mathcal{X}(\Sigma) \] , \[ \mathcal{X}(M_2) \in \mathcal{X}(\Sigma)^* \]

\[ \mathcal{Z}(M) = \langle \mathcal{Z}(M_1) | \mathcal{Z}(M_2) \rangle \]

(Atiyah's topological quantum field theory)
But $\mathcal{X}(\Sigma)$ = the space of all sections is too large. As is common in quantization, we should pick up a smaller space by choosing a “polarization”.

Pick a complex structure $J$ on $\Sigma$.

Then

$$\mathcal{A}_\Sigma \cong \Omega^1(\Sigma, \mathbb{G}) \cong \Omega^0(\Sigma, \mathbb{G} \otimes \mathbb{C}) \subset (\infty - \text{dim} \mathbb{C}) \text{ cpx mod}$$

Moreover $\mathcal{G}^\mathbb{C}_\Sigma = \text{Map}(\Sigma, \mathbb{G}^\mathbb{C})$ : cpxification of $\mathcal{G}_\Sigma$ acts on $\mathcal{A}_\Sigma$ holomorphically by

$$A^i \mapsto g^i \tilde{g} + g^{-1}A^i g$$

Also $\mathcal{Z}$ has a natural fiber structure.

Then it is natural to put

$$\mathcal{X}(\Sigma) = \text{space of holomorphic sections of } \mathcal{L}^\otimes \text{ on } \mathcal{A}_\Sigma/\mathcal{G}^\mathbb{C}_\Sigma$$

Comment:

We could also consider an intermediate vector space of sections of the symplectic quotient $\mu^\ast(0)/\mathcal{G}_\Sigma = \text{moduli of flat connections on } \Sigma$. The above is its geometric quantization.
\[ \mathcal{M}_{G_c} = \text{moduli stack of } G_c \text{-bundles on } (\Sigma, J) \]

So \[ H^0(\mathcal{L}^{\otimes g}) = \text{the space of conformal blocks} \]

When \[ L \subset M, \{ p_1, \ldots, p_L \} = L \cap \Sigma \]

\[ \mathcal{Z}_{G_c, R_1, \ldots, R_L}(\Sigma, p_1, \ldots, p_L) \]

\[ = \text{the space of holomorphic sections of the line bundle} \]
\[ \text{moduli stack of parabolic } G_c \text{-bundles on } (\Sigma, J) \]

i.e., \[ G_c \text{-bundle together with reduction} \]
\[ \text{of } G_c \to \text{Borel at each marked point} \]

The line bundle \[ L \] now depends also on \[ R_i \]

\[ = \text{the space of conformal blocks attached to} \]

the data \[ R_1 \]
\[ \vdots \]
\[ R_L \]

\[ (\Sigma, J) \]
There is one thing to be checked:
\( \Xi(\Sigma) \) "should" be a topological invariant.

\( \exists \) (projectively) flat connection on the bundle of conformal blocks over the moduli space of pointed Riemann surfaces.

**Perturbation theory**

Suppose \( A \) is a flat connection:

\[
\text{CS}(A + \alpha) = \text{CS}(A) + \frac{1}{8\pi^2} \int_M \text{Tr}(\alpha \wedge d_A \alpha + \frac{2}{3} \alpha \wedge dA \wedge \alpha)
\]

- stationary phase approximation
  - (ignore the cubic term)

\[
\frac{\Xi(\Sigma)}{\Xi} \sim \text{large} \sum_{[A] : \text{flat connection}} \alpha([A]) e^{2\pi i \text{CS}(A)}
\]

Remark. In general, the moduli space of flat connections are not isolated points, not even a smooth manifold.

But we ignore this point, and assume \( H^*_F = 0 \).
Recall $\int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}}$

By analytic continuation $\int_{-\infty}^{\infty} e^{i\lambda x^2} \, dx = \frac{1}{\sqrt{|\lambda|}} \exp\left(\frac{\pi i}{4} \text{sgn} \lambda\right)$

$\lambda \in \mathbb{R}$

For a quadratic form $Q$ on $\mathbb{R}^n$

$\int_{\mathbb{R}^n} e^{i\langle Q(x), x \rangle} \, dx \cdots dx_n = \frac{1}{\sqrt{\det Q}} \exp\left(\frac{\pi i}{4} \text{sgn} Q\right)$

In our case we want to apply this formula to $\mathbb{R}^n$

$\mathbf{c} \mapsto T_{\mathbb{C}}(\mathbf{A}/g)$

$Q \mapsto \frac{1}{4\pi} \sum_{x} \text{Tr}(\mathbf{a} \wedge d\mathbf{a})$

$0 \in T_{\mathbb{C}}(\mathbb{A}/g)$

We take a "slice" to the gauge group orbit standard recipe:

$\mathbf{A}$

Pick up a Riemannian metric $g$ on $M^3$, and consider

$g \cdot A$

$\text{Ker}(d_A : \Omega^1(M) \otimes g \rightarrow \Omega^0(M) \otimes g) \equiv T_{\mathbb{C}}(\mathbf{A}/g)$
We also need to understand the Jacobian of
\( Kn d^* \rightarrow A^s / A^s \) to compare the Feynman measure.

Then finally (see [Atiyah] for more details)
\( \det Q, \text{sgn } Q \) are expressed in terms of \( \Delta_A^{(c)} \) : laplacian on \( \Omega^c(M; \mathfrak{g}) \)
\& \( DA = (dA + * dA^*) : \Omega^{\text{odd}} \rightarrow \Omega^{\text{even}} = \Omega^{\text{odd}} \)

Answer
\[
\sqrt{\text{det } RQ} = \left( \frac{\text{det } \Delta_A^{(c)}}{\text{det } \Delta_A^{(p)}} \right)^{1/4}
\]

\( \text{sgn } Q = \text{sgn } DA \)

Now we use the Ray-Singer \( \zeta \) function regularization to define \( \det \Delta_A^{(c)} , \text{sgn } DA \)

\[
\zeta_A(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s} \quad \lambda \text{ : eigenvalue of } \Delta_A^{(c)}
\]

\[
\zeta_A(0) = \text{dim} \Omega^c \quad = 0 \quad \text{in odd dim}
\]

\[
\exp(-\zeta_A(0)) = \quad \text{det} \Delta_A
\]
i.e. in our case
\[ \eta(s) = \sum_{\lambda \neq 0} |\lambda|^{-s} \text{sign} \lambda \quad \text{for } \lambda \in \mathbb{R} \]

\[ \eta(0) = \text{"sgn" } D_A \]

The (Cheeger, Müller)

\[ \frac{(\det \Delta_A^{(0)})^{3/2}}{(\det \Delta_A^{(0)})^{1/2}} = \text{Reidemeister torsion} \]

metric independent!

The phase factor is more subtle, as \( \eta(0) \) is not a topological invariant.

The invariant depends on the choice of the "framing" of \( M \), i.e., \( TM \cong M \times \mathbb{R}^3 \) trivialisations

framing of link

\( \xrightarrow{\text{string}} \) ribbon
O Perturbation theory.
- finite dim'l model

\[ \mathcal{X}_\Delta := \int_{\mathbb{R}^n} dx \exp \left[ i \mathcal{R} (Q(x) + T(x)) \right] \]

\[ T \text{: cubic form} \]

\[ = \int_{\mathbb{R}^n} dx \exp (i \mathcal{R} Q(x)) \sum_{m=0}^{\infty} \frac{1}{m!} \left( i \mathcal{R} T(x) \right)^m \]

We introduce a new variable \( u \in \mathbb{R}^n \) (auxiliary field)

\[ \frac{\partial}{\partial u_a} \exp (i <u,x>) \bigg|_{u=0} = ix_a \]

\[ \frac{\partial^2}{\partial u_a \partial u_b} \exp (i <u,x>) \bigg|_{u=0} = ix_a ix_b , \ldots \]

\[ \therefore T(x)^m = \left( \sum_{a,b,c=1}^{n} T_{abc} x_a x_b x_c \right)^m \]

\[ = \frac{1}{i^m} \left( \sum_{a,b,c=1}^{n} T_{abc} \frac{\partial^m}{\partial u_a \partial u_b \partial u_c} \right)^m \exp (i <u,x>) \bigg|_{u=0} \]

We put \( \exp (i <u,x>) \) to \( \exp (i \mathcal{R} Q(x)) \exp (i \mathcal{R} Q(x) + i <u,x>) \)

Complete the square:

\[ \exp \left[ i \mathcal{R} Q(x') - \frac{i}{4 \mathcal{R}} <u, Q u> \right] \quad x' = x + \frac{1}{4 \mathcal{R}} Q^{-1} u \]
Our measure is translation invariant. (the same is true for the Feynman measure)

\[ Z_a = \frac{C}{\sqrt{\text{det} Q}} \exp \left( \frac{i\pi}{4} \text{sgn} Q \right) \]

\[ \times \sum_{m=3}^{\infty} (-1)^m \frac{\partial^m}{\partial u^m} \left( \sum_{a,b,c} \frac{1}{4\pi k} \exp \left( -\frac{i}{4\pi} \langle u, Q u \rangle \right) \right)^m \]

We can expand the second part: \[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^m}{4k} \langle u, Q u \rangle^n \]

Simplest term

\( m=2, n=3 \)

\[ \text{Tr} \bar{c}ab^c Q^{ab} Q^{bc} Q^{cc} \quad \text{&} \quad \text{Tr} \bar{c}ab^c Q^{ab} Q^{bc} Q^{cc} \]

(\( \overline{Q} \) permutation of indices)

\[ \text{graphically} \]

We put \( T \) at vertex, \( m=\# \text{vertex} \)

\( Q \) edge \( n=\# \text{edge} \)

\( m-n = e(\Gamma) \)

(Feynman graph)
We get \( \exp \left( \sum_{P: \text{connected graph}} \frac{e(P) \chi(P)}{|\text{Aut}(P)|} \right) \).

\( \chi(P) \) is defined as above.

We apply this argument to the co-dim\( \ell \) setting:

We represent \( Q^{-1} = \sum_{M} L(\cdots, x) \times (\infty) \)

i.e. \( Q^{-1} = L^* \)

\( \rightarrow \) \( \chi(P) \) is given by an integration over \( \underbrace{M \times \cdots \times M}_{2m} \)

Remark, \( P \) is shifted by \( \mathbb{R} + \mathbb{Z}^V \)

dual Coxeter number (quantum correction)

\[
\left( \varphi_{\mathfrak{a}(0)} - \varphi_{\text{triv.}(0)} \times \mathbb{Z}^V \cdot \text{CS}(A) \right)
\]
Comment:
The perturbative invariants can be proved to be independent of a Riemannian metric. They give contributions of a flat connection (or a component of the moduli space of flat connections).

However, the exact invariant is well-defined only for an integer $k$. It is probably not possible to single out the contribution of a flat connection for a general 3-manifold.

Q: Do you understand why link invariants in $S^3$ can be defined for arbitrary $p$-shaun, not necessarily roots of unity?

A: No, except by computation.