Perverse coherent sheaves on a blowup surface II

\( X \) : nonsingular projective surface / \( \mathbb{C} \)

\( x \in X \)

\( p: \hat{X} \rightarrow X \) blowup at \( p \)

\( \mathcal{O} \rightarrow p \)

Bridgeland considered more general setting

- birational \(- R_{p*}(\mathcal{O}_{\hat{X}}) = \mathcal{O}_X \)

- relative dim. one

\( \mathcal{A} = D(\hat{X}) \) : derived category of coherent sheaves on \( \hat{X} \)

\( \mathcal{B} = D(\hat{X}) \xrightarrow{p^*} D(X) \) full subcategory

\( \mathcal{C} = \mathcal{B}^{-1} = \{ a \in \mathcal{A} | \text{Hom}_X(b, a) = 0 \text{ for } b \in \mathcal{B} \} \)

\( \mathcal{A} = \langle \mathcal{C}, \mathcal{B} \rangle \)

\( \Rightarrow \text{Perv}(\hat{X}/X) \subset D(\hat{X}) \) perverse coherent sheaf obtained by "gluing" cores of \( \mathcal{B} \) & \( \mathcal{C} \) in a different way

Def. (Bridgeland)

\( E \in D(\hat{X}) \) is perverse coherent (\( \in \text{Perv}(\hat{X}/X) \))

(i) \( H^{i}(E) = 0 \) for \( i \neq -1, 0 \)
(ii) \( p_*(H^q(E)) = 0 \), \( R^1 p_*(H^0(E)) = 0 \)

(iii) \( \text{Hom}( H^0(E), c) = 0 \) \( \forall c \in \mathcal{C} \cap \text{Coh} \hat{X} \)

- \( R^p E \in \text{Coh} \hat{X} \) Thus \( \text{Peru}(\hat{X}/X) \) is close to \( \text{Coh} X \).
- \( \text{Peru}(\hat{X}/X) \) is an abelian category.

Example

\[
\begin{align*}
\gamma \in \mathcal{C} & \quad 0 \to \mathcal{O}(c) \to J_y \to \mathcal{O}_c(-1) \to 0 \\
\text{Peru}(\hat{X}/X) & \quad \text{Peru}(\hat{X}/X)[1] \\
\text{But} & \quad J_y \notin \text{Peru}(\hat{X}/X)
\end{align*}
\]

Exchange

\[
\begin{align*}
0 & \to \mathcal{O}_c(-1) \to E \to \mathcal{O}_c(c) \to 0 \quad (\star) \\
L, R & \quad \text{or} \quad 0 \to E \to \mathcal{O}_c(c) \to \mathcal{O}_c(-1)[1] \to 0 \\
& \quad \text{Peru}(\hat{X}/X)[1]
\end{align*}
\]

Since \( \dim \text{Ext}^1(\mathcal{O}_c(c), \mathcal{O}_c(-1)) = 1 \), \( \star \) is unique up to isomorphism

\[
\therefore \text{Moduli of Peru. coh} \text{"ideal" sheaves } \cong \hat{X}
\]

Revis.
Bridgeland considered 3-dim? situation
- Moduli of Peru. coh "ideal" sheaves \( \cong X^+ \) flops
- \( D(X) \cong D(X^+) \) given by FM transform
  \( \text{wrt. the universal family} \)
Thus it is natural to expect moduli spaces of perverse coherent sheaves on $X$ should be close to moduli spaces of coherent sheaves on $X$.

More precisely, put the "stability" condition so that the above is true.
\( H \): ample line bundle on \( X \) 

**Assume** \( H \)-stable \( \iff \) \( H \)-semistable

**Def.** \( E \in \text{Perv}(\mathcal{X}/X) \) is stable

1. \( E \in \text{Coh} \mathcal{X} \) (i.e. \( H^{-}(E) = 0 \))
2. \( p_{*}E \) is an \( H \)-stable torsion-free sheaf.

**Rem.** This, in fact, comes from more conceptual definition via comparison of "Hilbert" polynomials on subobjects.

**Little bit more tractable definition**

**Stable**

1. \( \text{Hom}(\Theta_{c}, E) = 0 \) \( \iff \) torsion-freeness of \( p_{*}E \)
2. \( \text{Hom}(E, \Theta_{c}(-1)) = 0 \) perverse cond. (iii)
3. \( p_{*}E \) is \( H \)-stable

Under the wall-crossing 1), 2) are violated.
$M^p_H(c) =$ moduli space of stable perverse coherent sheaves on $\hat{X}$

$M^c_m(c) := \{ E \in \text{Gr}^X \mid E(-mC) \in M^p_H(c) \}$

$\begin{align*}
(C, c) = 0 & \quad M^0(c) & M^1(c) & \cdots & M^N(c) \\
\text{Moduli on } X & \quad \text{Moduli on } \hat{X}
\end{align*}$

0 1st assertion is not so difficult:

Lemma: $M^0(c) \cong M^p_H(c) \cong M^p_{X,H}(c)$ if $(C, c) = 0$

$E \mapsto p^*_p \phi$ (sketch)

Consider $p^*_p \phi \phi \mapsto E$. Perverse $\Rightarrow$ surjective

A little bit of thought $\Rightarrow \text{ker} \phi \cong \mathcal{O}_C(\leq p)^{\oplus p} \quad (p \geq 0)$

Chern class cond. $\Rightarrow p = 0$
Two constructions of moduli spaces:

1) Following Bridgeland, take the quotient of $\text{Quot}$ wrt $p^*Q_x(1)$.

2) Consider framed sheaves on $\mathbb{P}^2$, $\hat{\mathbb{P}}^2$, and use monad description (of $\text{King}$).

Adv. of 1)
- Works for any surface

Disadv. of 1)
- The usual moduli & perverse moduli cannot be constructed simultaneously.
  Only individual $M_{g(c)}$.  

2) \((E, \Phi)\) : framed sheaf on \(E^2 = \mathbb{C}^2 \cup \mathbb{C}^\infty\)

\[ \Phi : E|_{\mathbb{C}^2} \cong \mathbb{C}^\infty \]

framed moduli:
\[
\begin{cases}
\mathbb{C} V \mathcal{U} & a) \left[ B_1, B_2 \right] + i\gamma = 0 \\
\mathbb{C} W & b) S \subset \mathcal{U} \text{ s.t.} \\
\text{Im} \gamma \in \mathbb{C}, \ B_0(S) \subset \mathbb{C}
\end{cases}
\]

where \(\mathcal{U}, W\) : vector spaces of \(\dim = \text{deg}(E), \ \text{rank} E\)

0 variant on \(E^2\)

\[
\begin{array}{ccc}
\mathcal{T}_0 & \xrightarrow{\delta} & \mathcal{U}_1 \\
\mathcal{U}_0 \xleftarrow{\delta} & \xrightarrow{\iota} & \mathcal{U}_1 \\
W \xleftarrow{\delta} & \xrightarrow{i} & \mathcal{U}_0 \\
B_1 d B_2 - B_2 d B_1 + i\gamma = 0 & \text{dim} \mathcal{T}_0 = \text{dim} \mathcal{T}_1 = 0_2 \\
\text{dim} W = \text{rank} E
\end{array}
\]

Taking GIT quotient w.r.t. the trivial line bundle with nontrivial action \(\text{GL}(\mathcal{T}_0) \times \text{GL}(\mathcal{T}_1) \xrightarrow{X} \mathbb{C}^\ast\)

\[X = X_{S_0, S_1} = (\det \gamma_0)^{S_0} \cdot (\det \gamma_1)^{S_1}\]

We can construct the moduli for param. on the wall.
Wall-crossing and coherent systems

Suppose \( E \in \text{M}^+ \) and \( E \in \text{M}^+_{\text{st}} \)

\[
E \in \text{M}^+ \implies \text{Hom}(E, \mathcal{O}_C(-m-1)) = 0 \quad \text{by definition}
\]

(\( \text{Hom}(\mathcal{O}_C(-m), E) = 0 \))

\( E+ \in \text{M}^+_{\text{st}} \implies \text{Hom}(\mathcal{O}_C(-m-1), E+) = 0 \) (This should be violated!)

(\( \text{Hom}(E+, \mathcal{O}_C(-m-2)) = 0 \))

Then

\[
0 \to \mathcal{O}_C(-m-1)^\oplus \to E_+ \to E' \to 0 \quad \text{with} \quad E' \in \text{M}_m \cap \text{M}_{\text{st}}(c')
\]

(JH filtration w.r.t. the stab. param. on the wall)

Conversely, if \( E+ \in \text{M}^+_{\text{st}} \) and \( E+ \in \text{M}^+ \)

Then

\[
0 \to E' \to E_+ \to \mathcal{O}_C(-m-1)^\oplus \to 0 \quad \text{with} \quad E' \in \text{M}_m \cap \text{M}_{\text{st}}(c')
\]

This is a typical picture in the wall-crossing

L term \( \leftrightarrow \) R term in a short exact seq.
Brill-Noether locus

\[ M^p_m = \{ E_- \in M_- \mid \text{dim } \text{Hom}(\mathcal{O}_C(-m-1), E_-) = p \} \]

\[ M^p_{m+1} = \{ E_+ \in M_+ \mid \text{dim } \text{Hom}(E_+, \mathcal{O}_C(-m-1)) = p \} \]

Thus under the wall-crossing:

\[ M^p_m \leftarrow \bigcup_{p>0} M^p_m \rightarrow M^p_{m+1} \]

\[ M^p_m \cap \bigcup_{p>0} M^p_m = M^p_{m+1} \cap \bigcup_{p>0} M^p_{m+1} \]

Further, \( M^p_m \) is a Grassmann bundle of \( p \)-planes in \( \text{Ext}^1(E', \mathcal{O}_C(-m-1)) \) and \( M^p_m \cap M^p_{m+1}(c') = M^p_m(c') \)

Similarly, \( M^p_{m+1} = \text{Gr}(p, \text{Ext}^1(\mathcal{O}_C(-m-1), E')) \).

By dimension calculation:

\[ M^p_m = \bigcup_{q \geq p} M^q_m \quad \text{singular} \]

Rem. Ass. is not true

for \((-2)\)-curve case \( \Rightarrow \) One of key ingredients of the geometric construction of Kashiwara crystal.
Note \( \text{Ext}^1(E', \mathcal{O}_C(-u-1)) \) extends to \( M_m(c) \)
(but not to \( M_{m+1}(c) \))

\[
\begin{array}{c}
\text{Gr}(p, \text{Ext}^1(E', \mathcal{O}_C(-u-1))) \rightarrow M^\leq_{m+1}(c) \\
\downarrow \\
M_m(c')
\end{array}
\]

resolution of singularities

Using this diagram, the wall-crossing formula of the integration can be done recursively.
It is algorithmic, but the final expression is difficult to handle so far.....