2007/11/16 Summary

1. CS
2. open GW
3. closed GW
4. $U(N)$-instanton counting on $\mathbb{R}^4$

$S^3 \quad T^*S^3 \quad S^3 \quad \mathcal{O}(1) \otimes \mathcal{O}(1)$

$N, \mathcal{Q}, q_s, t \quad q_s, Q \quad q_s, Q$

$\frac{2\pi i}{\mathcal{Q}} = q_s, \quad \frac{2\pi i N}{\mathcal{Q}} = t, \quad Q = e^{-t}$

$(\delta (\text{quantum group param.}) = e^{2})$

1. SU($N$) CS partition function of $M^3$, level $\mathcal{Q}$

$Z_{\text{SU($N$), CS($M^3$)}} = \int_{\mathcal{A}/G} e^{2\pi i \mathcal{Q} \text{CS($\mathcal{A}$)}} \text{DA}$

1’ exact solution $\leftrightarrow$ Both are defined rigorously.

1” perturbation theory $\leftrightarrow$

$Z_{\text{SU($N$), CS($M$)}} \sim \text{(stationary phase)} \times \exp \left[ \sum_{r=1}^{\infty} \mathcal{S}_r \left( \frac{2\pi i \mathcal{Q}}{\mathcal{Q} + N} \right)^r \right]$

or 1-loop

2. open GW invariants $T^*M^3$ with lagrangian $M^3$

$\log \mathcal{X}(S^3) = \sum_{p=0}^{\infty} q_s^{2g-2} \# \{ f: (\Sigma, \partial \Sigma) \to (T^*S^3, S^3) \}$

$p = \# \partial \Sigma$

$g = \text{genus}$

So far, not defined rigorously

3. closed GW invariants of $(\mathcal{O}(1) \otimes \mathcal{O}(1)) \to \mathbb{R}^1$

$\log \mathcal{X(\text{resolved, conifold})} = \sum_{g, d} q_s^{2g-2} Q^d \# \{ f: \Sigma_g \to \text{resolved} \}$

$d \geq 1$

$\mathcal{E}$ defined rigorously
\[ U(1) \text{-instanton counting (K-theory)} \]

\[ C^2 \subset C^\ast \quad (e^{i\theta x}, e^{-i\theta y}) \]

\[ X = \sum_{k \geq 0} \text{ch} (\mathcal{M}(k, \Omega)) Q^k \quad \text{defined rigorously} \]

**Dualities:**

(1, 2, 3, 4) are all equal.

(M = S^3) (except possibly 2)

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In the simplest case (i.e. M = S^3 without link)

we can explicitly compute

\[ \mathcal{Z}_{\text{super}}(S^3) = \text{SO}_0 \quad (S\text{-matrix in CFT}) \]

\[ \quad \exp\left(-\sum_{n=1}^{\infty} \frac{Q^n}{n(e^{\frac{2\pi}{n\theta}} - e^{-\frac{2\pi}{n\theta}})}\right) \]

We can check they are equal (modulo perturb. terms) by direct computation.

Sometimes we need an analytic continuation.

**CS:** N: fix \( R \): large

\( g_s \): small, \( \tau = N g_s \): small

**closed GW:** \( g_s \): small, \( Q = e^{-\tau} \): small

**instanton**
Identification of invariants by physical intuitions:

\( 0' = 0'' \) : perturbation expansion

not rigorous so far, but if we can expand the path integral as in the fin. dim. integral, it is ok.

\( 0'' = 2\) (Witten) string field theory

(This is rather difficult.)

\( \equiv \) mathematical framework by Fukaya even for physicists.

\[ S_r = \sum_{\Gamma} a_{\Gamma}(SU(N)) \cdot I_{\Gamma}(M) \]

\( \Gamma \): trivalent graph connected with 2r vertex

\( a_{\Gamma} \): (Killing form)\(^{-1}\) \( I_{\Gamma} \)

\[ L = d^* \Delta^{-1} \]

idea: we are at “degenerate” situations

Let’s perturb propagator/lagrangian

by Morse functions!
\[ \text{propagator: } \Delta \mapsto e^{\frac{1}{\varepsilon} \Delta} e^{-\frac{1}{\varepsilon} f} \]

\[ \text{vs. } \]

\[ \text{lagrangian } M \mapsto \text{graph of } E_\varepsilon \frac{f}{\varepsilon} \]

Since these are topological theories, invariants are independent of \( \varepsilon \).

\[ \text{CS: } \varepsilon \to \infty \text{ original (Morse flow)} \]

\[ \varepsilon \to 0 \text{ counting of gradient graphs} \]

\[ \text{lagrangian: } \varepsilon \to 0 \text{ Riemann surfaces become thin.} \]

\[ \to \text{gradient graphs} \]

\[ \therefore \]

\[ 't \text{Hooft "filling role"} \]

\[ \frac{\partial g_2}{\partial s} \]

\[ \text{fix de sum up over } \Phi \]

\[ \text{geometric transition} \]

\[ \text{T}*S^3 \mapsto 2y = 8w \leftarrow \text{O(4)} \otimes \Omega(4) \]

\[ \text{deform } \]

\[ \text{resolv. } \]

\[ \text{ } \]

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\[ \text{ } \]
This geometric explanation is very appealing, but note that we need to make change of variables as
\[ e^{-t} = Q \]

\[ t: \text{small in open GW} \quad Q: \text{small in closed GW}. \]

Andrew's comment: Ooguri-Vafa

Paper: /0205297
geometric engineering

In this case, both invariants are exactly equal. Moreover combinatorial expressions are the same. Thus geometric picture is not clear in this duality, but computational relation is VERY strong.

GW side

\[ \phi \prod \phi = \sum_{\lambda} \phi^{\lambda} \times \phi^{(-Q)^{\lambda}} \]
(in character basis)

• instanton counting

localization formula

\[ H\text{ilb}^n \mathbb{C}^2 \rightarrow S^n \mathbb{C}^2 \]
resolution

\[ \text{ch}(\mathbb{G}^n \mathbb{C}^2) = \text{ch}(H^* \text{H}(\text{Hilb}^n \mathbb{C}^2)) \]

\[ = \sum_{\alpha} \frac{1}{\text{ch} \wedge_{-T^*_X} \text{Hilb}^n \mathbb{C}^2} \]

\[ T^*_X \text{Hilb}^n \mathbb{C}^2 = \sum_{\alpha} e^{q\alpha} e^{-q\alpha} \]

\[ \text{stable} \quad \text{unstable} \]

\[ \text{mfd} \quad \text{mfd} \]
One more example of analytic continuation
\[
\begin{align*}
\chi_{\mu \nu \lambda}(g_s, Q) & \sim \chi_{\mu \nu \lambda}(g_s, Q') \quad \text{if} \quad Q = Q^{-1} \\
\end{align*}
\]

Remark: The "Chain" of dualities still continue......

\( \circ \): \( A \)-model \( \leftrightarrow \) \( B \)-model

\[ \quad \text{mirror} \]

We discuss genus 0 part in Dec.
\[ \rightarrow \text{SW curve.} \]

\( \circ \): GW v.s. Gopakumar-Vafa invariants (BPS counting)
\[ \quad \text{or Donaldson-Thomas invariants} \]

\( \circ \): CS perturbation theory
\[ \rightarrow / \text{Rozansky-Witten invariants} \]
\[ \quad / \text{finite type invariants} \]

For the proof of the topological invariance of
\[ S_T = \sum \alpha_P(g) \text{I}_P(M^3), \]
we only need the IHX relation on $\mathcal{A}_p(\Sigma)$

\[
\begin{array}{c}
\pi(x,y) = \pi(x,y) - \pi(y,x)
\end{array}
\]

\[
[[x,y],z] = [x,[y,z]] - [y,[x,z]]
\]

(Jacobi identity)

If we have an assignment $\Gamma \mapsto \mathcal{A}_p$ satisfying this relation, we get a topological invariant.

Such an assignment was given by Rozansky–Witten for a given hyperKähler manifold $(X,\omega)$.

\[\text{put a curvature } R_{\mathbb{C}} \]

\[\text{holo. symplect. form} \]

\[\text{get } \tilde{\mathcal{A}}_p(X) \in \Omega^{0,2r}(X) \]

\[\sum_X \omega^r \tilde{\mathcal{A}}_p(X) = \mathcal{A}_p(X) \]

IHX rel. follows from the Bianchi identity.

Universal invariant

\[
(\text{trivalent graph } / \text{IHX})^4 \Rightarrow \Gamma \mapsto \mathcal{A}_p \in \mathcal{R}_5
\]
I said invariants are all equal. But this statement is not quite true.

Disagreements come from the constant map contributions to the GW invariants

\[ \text{Mg}_0(X) \cong X \times \text{Mg}_0 \]

\[ \uparrow \]

\[ \Xi_{\text{nonopt}} = \text{e}(T_x) \cdot \text{c} \cdot \int_{\text{Mg}_0} g \cdot (\text{e})^3 \]

If \( X \) would be compact, \( \text{e}(T_x) \cdot (X) = \text{e}(X) \).

In our case we should formally define the constant map contribution by the same formula.

There is still mismatch in genus 0, 1

( In these cases \( \text{Mg}_0 = \emptyset \) )

But certainly there should be something, if the invariants are defined via the path integral. It should be possible to compute them via the perturbative expansion?
### Generalization / Variants

- Framing
  - choice of trivialization of the tangent bundle of $M^3$

2. ?
3. normal bundle of $\mathbb{P}^1 \otimes \mathbb{C}^{n+1}$ : sta
   $\to \mathbb{C}^n \otimes \mathbb{C}^{n-2}$
4. twist by the line bundle $L = \det \mathbb{O}_2$
   where $\mathbb{O}_2$ fibers at $I = \mathbb{C} \times \mathbb{C}$

### Link

1. $SU(N)$ CS partition function of $(S^3, L)$
   level $R \in \mathbb{Z}$
   
   $Z_{R; R_1; \ldots; R_k}(S^3, L) = \sum_{\text{tr} H^0 L_i(A)} e^{2\pi i R \text{CS}(A)} \text{tr} H^0 L_i(A) \; \text{DA}$

2. exact solution
   
   Hilbert space = conformal blocks
   
   e.g. $Z_{R}(S^3) = \text{SO}_n$, $Z_{R; R} (\text{unknot}) = \text{SO}_R$,
   $Z_{R; R; R}(\bigcirc) = SR; R$

3. Perturbation theory
   
   $Z_{R; R}(S^3, L) \sim \frac{1}{R}$ as $R \to \infty$
$\partial$ $\mathcal{C}_L$ : canonical bundle to $L_i \subset T^*S^3$
open GW inv. of $T^*S^3$ with $\bigcup_i \mathcal{C}_L$

$\partial$ $\mathcal{C}_L_i$ : transition of $\mathcal{C}_L$

$\partial$ Sergei : $U(1)$ instanton with singularities along axis
$N :$ vector bundles over $\text{Hilb}^n$
( on cpx )

$\Gamma \subset SU(2)$

$\Gamma :$ CS partition functions on $S^3/\Gamma$

[Note: flat connection $\sim \bigoplus \rho_\Gamma \otimes N_i$ $\rho_i :$ ir. rep.]

$\partial$?

$\partial$ ( $O(1) \oplus O(1)) / \Gamma$

( how about different model ? )

$\partial$ gauge theory partition function for $G = G(\Gamma)$ via McKay
$N_i \leftrightarrow a_i$