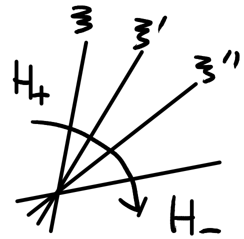
REVIEW OF THE 1^{ST} LECTURE

- (generalised) Donaldson invariant

$$\Phi_H^{r,c_1}\left(\exp\sum_{p=1}^{\infty}\alpha_p\right) := \sum_c \Lambda^{\dim M_H(c)} \int_{M_H(c)} \exp\left(\sum_{p=1}^{\infty} \mu_p(\alpha_p)\right) \qquad (c_1, r : \text{fix})$$

- Theorem (Mochizuki) (rank = 2)

$$\Phi_{H_+}^{c_1,2}\left(\exp\sum\alpha_p\right) - \Phi_{H_-}^{c_1,2}\left(\exp\sum\alpha_p\right) = \sum_{\xi\in B_+} \delta_\xi^X\left(\exp\sum\alpha_p\right)$$



where δ_ξ^X = coeff of t^{-1} in $\delta_{\xi,t}^X$

$$\delta_{\xi,t}^X := \sum_{\ell\geq 0} \Lambda^{4\ell-\xi^2-3} \int_{X_2^{[\ell]}} \frac{\exp\left(\sum (-1)^p \left[ch(\mathcal{I}_1)e^{\frac{\xi-t}{2}} + ch(\mathcal{I}_2)e^{-\frac{\xi-t}{2}}\right]_{p+1} / \alpha_p\right)}{e(-\mathrm{Ext}_p^\bullet(\mathcal{I}_2, \mathcal{I}_1(\xi))e^{t})\, e(-\mathrm{Ext}_p^\bullet(\mathcal{I}_1(\xi), \mathcal{I}_2)e^{-t})}$$

- Theorem (Ellingsrud – Göttsche – Lehn, GNY)

$\exists$ universal $A_1, A_2, \cdots A_8^p$

$$(-1)^{\chi(\mathcal{O}_X)+\xi(\xi-K_X)/2}\, t^{-\xi^2-2\chi(\mathcal{O}_X)}\, \Lambda^{\xi^2+3\chi(\mathcal{O}_X)}\, \delta_{\xi,t}^X$$

$$= \exp\Big[A_1 \cdot \int_X \xi^2 + A_2 \int_X c_1(X)\cdot\xi + A_3 \int_X c_1(X)^2 + A_4 \int_X c_2(X)$$

$$+ \sum_p A_5^p \int_X \xi\cdot\alpha_p + A_6^p \int_X c_1(X)\cdot\alpha_p + A_7^{p,p'} \int_X \alpha_p\alpha_{p'} + A_8^p \int_X \alpha_p\Big]$$

- X : toric surface $\quad X^T = \{p_1, \cdots, p_\chi\}$

$$\tilde{\delta}_{\xi,t}^X(\text{equivariant lift}) = \prod_{i=1}^{\chi} \delta_{\xi|_{p_i},t}^{X=\mathbb{C}^2} \quad (\text{product of local contributions at } p_i)$$

THIS IS THE INSTANTON COUNTING PARTITION FUNCTION.

PLAN OF THE 2ND LECTURE

PART II. Instanton Counting

2.1. Definition

2.2. Seiberg–Witten curve

mirror of the Donaldson invariants on $\mathbb{C}^2 = \mathbb{R}^4$

2.3. Computation of Wall-crossing terms

2.4. A DETOUR : Geometric Engineering

~~2.5. A conjectural link to Khovanov homology~~

PART 2. Instanton counting

2.1. Definition

$X = \mathbb{C}^2 = \mathbb{R}^4$

$M_0^{reg}(n,r)$: framed moduli space of $U(r)$-anti-self-dual connections on $\mathbb{R}^4$ with $c_2 = n$

$$= \left\{ A : G\text{-connection on } \underset{\mathbb{R}^4 \cup \{\infty\}}{\overset{\shortparallel}{S^4}} \;\middle|\; \begin{array}{l} *F_A = -F_A \\ \frac{1}{8\pi^2}\int_{S^4} |F_A|^2 = n < \infty \end{array} \right\} \Big/ \underset{\gamma \text{ s.t. } \gamma(\infty) = id}{\text{gauge transf}}$$

C^∞-mfd of $\dim_{\mathbb{R}} = 4nr$ noncompact

Kobayashi-Hitchin correspondence :

$M_0^{reg}(n,r)$ = framed moduli space of locally free sheaves on $\mathbb{P}^2 = \mathbb{C}^2 \cup l_\infty$

$$= \left\{ (E, \Phi) \;\middle|\; \begin{array}{l} E : \text{locally free sheaf}, \ rk = r, \ c_2 = n \\ \Phi : E|_{l_\infty} \cong \mathcal{O}_{l_\infty}^{\oplus r} \end{array} \right\} \Big/ \text{isom.}$$

There are TWO partial compactifications.

- $M_0(n,r)$: Uhlenbeck partial compactification

$$= \coprod_{m=0}^{n} M_0^{reg}(n-m,r) \times S^m \mathbb{R}^4 \qquad S^m\mathbb{R}^4 = (\mathbb{R}^4)^m / \mathfrak{S}_m$$

- $M(n,r)$: Gieseker partial compactification

  = framed moduli space of torsion-free sheaves on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$

FACT. ① $M(n,r) \xrightarrow{\exists \pi} M_0(n,r)$ projective morphism

$(E,\Phi) \longmapsto (E^{\vee\vee},\Phi) \times \operatorname{Supp} E^{\vee\vee}/E$

② $M_0(n,r)$ is an affine algebraic variety

$M(n,r)$ is a smooth variety

$\tilde{T} := T^{r-1} \times T^2 \curvearrowright M(n,r)$ by $T^{r-1} \subset SL(r)$: change of framing
T^2 : via the action on $\mathbb{C}^2 = \mathbb{R}^4$

We consider the equivariant cohomology $H^*_{\tilde{T}}(M(n,r))$.

This is a module over $H^*_{\tilde{T}}(\text{point}) = \mathbb{C}[\varepsilon_1, \varepsilon_2, a_1, \dots, a_r]$

$(\mathrm{Lie}\, T^2 \otimes \mathbb{C} = \mathbb{C}[\varepsilon_1, \varepsilon_2]$, $\mathrm{Lie}\, T^{r-1} = \mathbb{C}[a_1, \dots, a_r]$

$a_1 + \cdots + a_r = 0$

Def. (Nekrasov instanton partition function)

$\vec{\alpha} = (\alpha_1, \alpha_2, \dots\dots)$

$$Z^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\alpha}; \Lambda) = \sum_{n=0}^{\infty} \Lambda^{2nr} \int_{M(n,r)} \exp\left(\sum_{p=1}^{\infty} (-1)^p \operatorname{ch}_{p+1}(\mathcal{E})/[\mathbb{C}^2] \cdot \alpha_p\right)$$

We will be mainly interested in the case $\alpha_p \equiv 0 \quad \forall p$

$$Z^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{n=0}^{\infty} \Lambda^{2nr} \int_{M(n,r)} 1$$

(<u>Rem</u>. If we are interested only in $p=1$ term (as in the ordinary Donaldson invariants),
α_1 can be absorbed in the change of $\Lambda \mapsto \Lambda e^{\alpha_1/4}$.

Definition of $/[\mathbb{C}^2]$ and $\int_{M(n,r)} \star$

Localization theorem in equivariant cohomology (Segal, Atiyah-Bott)

$\iota : X^T$ (= fixed pts) $\subset X$

$$H_T^*(X) \otimes_{H_T^*(pt)} \mathbb{C}(x_1,\dots,x_r) \xleftarrow[\cong]{\iota_*} H_T^*(X^T) \otimes_{H_T^*(pt)} \mathbb{C}(x_1,\dots,x_r)$$

$H_T^*(pt) = \mathbb{C}[x_1,\dots,x_r]$

Furthermore, if X is smooth, $(\iota_*)^{-1} = \sum_{\substack{X_\alpha \subset X^T \\ \text{component}}} \frac{\iota_\alpha^*}{e(N_{X_\alpha/X})}$ $\quad \iota_\alpha : X_\alpha \subset X$

In our cases X is noncompact, but X^T is compact.

So we can define $\int_X \star := p_*(\iota_*)^{-1}$ $\quad (p: X^T \to pt)$

Example. $\int_{\mathbb{C}^2} 1 = \frac{1}{\varepsilon_1 \varepsilon_2}$

Remark $\int_X$ has values in $\mathbb{C}(x_1,\dots,x_r)$, not in $\mathbb{C}[x_1,\dots,x_r]$.

rational function $\qquad$ polynomial

$$\therefore \int_{M(n,r)} \star = \sum_{\substack{x \in M(n,r)^{\tilde{T}} \\ \text{fixed point}}} \frac{\star|_x}{e(T_x M(n,r))}$$

Fixed points in $M(n,r)$

$$M(n,r)^{T^{r-1}} = \{ (E,\Phi) \mid E = \underset{\uparrow}{I_1} \oplus \cdots \oplus \underset{\uparrow}{I_r} \quad \Phi|_{I_\alpha} : I_\alpha \xrightarrow{\cong} \alpha^{\text{th}} \text{ factor } \mathcal{O}_{\ell_\infty} \text{ of } \mathcal{O}_{\ell_\infty}^{\oplus r} \}/\cong$$

ideal sheaf

$$= \coprod_{n_1 + \cdots + n_r = n} (\mathbb{C}^2)^{[n_1]} \times \cdots \times (\mathbb{C}^2)^{[n_r]} = (\underbrace{\mathbb{C}^2 \sqcup \cdots \sqcup \mathbb{C}^2}_{r \text{ terms}})^{[n]}$$

$$M(n,r)^{\tilde{T}} = \{ \vec{Y} = (Y_1, \cdots, Y_r) \mid r\text{-tuple of Young diagrams } |\vec{Y}| = \sum |Y_\alpha| = n \}$$

$e(T_{\vec{Y}} M(n,r))$ = product of $\tilde{T}$-weights of

$$T_{\vec{Y}} M(n,r) = \bigoplus_{\alpha,\beta} \mathrm{Ext}^1(I_\alpha, I_\beta(-\ell_\infty))$$

From this calculation, it is straightforward

$$\sum_{n=0}^{\infty} \Lambda^{2nr} \int_{M(n,r)} \exp\left[\sum (-1)^p \alpha_p \, ch_{p+1}(\mathcal{E})/_{[\mathbb{C}^2]}\right] = \tilde{\delta}^{X = \mathbb{C}^2}_{3,t} \quad \text{with } \alpha_p \rightsquigarrow \alpha_p[\mathbb{C}^2]$$

(for $r=2$)

Thus our remaining task is to compute $\tilde{\delta}_{3,t}^{X=\mathbb{C}^2}$ when $\varepsilon_1, \varepsilon_2 \longrightarrow 0$.

Set $\alpha_p = 0$.

Nekrasov Conjecture (Proved by N+Yoshioka, Nekrasov-Okounkov, Braverman-Etingof)

1) $\log Z^{\text{inst}} \in \frac{1}{\varepsilon_1 \varepsilon_2} \mathbb{C}[\varepsilon_1, \varepsilon_2, \vec{a}][[\Lambda]]$

2) $\varepsilon_1 \varepsilon_2 \log Z^{\text{inst}} \Big|_{\varepsilon_1=\varepsilon_2=0} = $ (the instanton part of) the Seiberg-Witten prepotential $\mathcal{F}^{SW}(\vec{a};\Lambda)$.

Remark. The partition function has the so-called perturbative part given by an explicit elementary function.

$$\varepsilon_1 \varepsilon_2 \log Z^{\text{inst}} + \text{pert. part} \Big|_{\varepsilon_1=\varepsilon_2=0} = \text{SW prepotential}$$

The Seiberg-Witten prepotential is defined by a period integral over a hyper-elliptic curve.

The conjecture is a kind of "mirror symmetry".

2.2. Seiberg-Witten curve

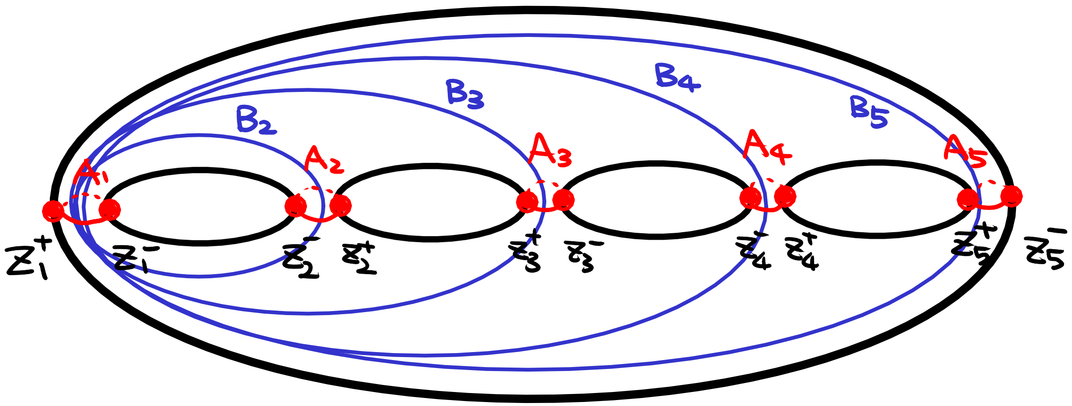
$\vec{u} = (u_2, \cdots, u_r) \in \mathbb{C}^{r-1}$

$C_{\vec{u}} : y^2 = P(z)^2 - 4\Lambda^{2r}$ $\qquad P(z) = z^r + u_2 z^{r-2} + \cdots + u_r$

$z_\alpha \ (\alpha = 1, \cdots, r)$: solutions of $P(z) = 0$

$z_\alpha^\pm$: solutions of $P(z) = \pm 2\Lambda^r$

s.t. $z_\alpha^\pm \approx z_\alpha$ $\qquad \Lambda$: very small



$\circlearrowleft$ hyperelliptic involution

Take A, B-cycles as above. $\left(\sum_{\alpha=1}^{r} A_\alpha = 0 \right)$

$\{A_\alpha, B_\alpha\}_{\alpha=2}^{r}$: symplectic base of $H_1(C_{\vec{u}})$

Seiberg-Witten meromorphic differential

$$dS := -\frac{1}{2\pi}\frac{z P'(z)\,dz}{y} \qquad (\text{poles at } \infty_\pm)$$

Define

$$a_\alpha \equiv a_\alpha(\vec{u}) = \int_{A_\alpha} dS \qquad (\alpha = 2, \cdots, r)$$
$$a_\alpha^D \equiv a_\alpha^D(\vec{u}) = \int_{B_\alpha} dS$$

$$\frac{\partial a_\alpha}{\partial u_p} = \frac{1}{2\pi}\int_{A_\alpha} \frac{z^{r-p}\,dz}{y}$$ a basis of holomorphic differentials

$\therefore (\tau_{\alpha\beta}) = \left(\frac{\partial a_\alpha}{\partial u_p}\right)^{-1}\left(\frac{\partial a_\beta^D}{\partial u_p}\right)$ is the period matrix of $C_{\vec{u}}$.

$= \left(\frac{\partial a_\beta^D}{\partial a_\alpha}\right) \longleftarrow$ symmetric!

$\therefore \exists \mathcal{F}$: potential s.t. $a_\alpha^D = -\frac{1}{2\pi\sqrt{-1}}\frac{\partial \mathcal{F}}{\partial a_\alpha}$ i.e. $\tau_{\alpha\beta} = -\frac{1}{2\pi\sqrt{-1}}\frac{\partial^2 \mathcal{F}}{\partial a_\alpha \partial a_\beta}$

This is the SW prepotential.

2.3. Computation of Wall-crossing terms

Let us go back to the wall-crossing term $\delta^X_{\mathfrak{z},t}$ for a toric X.

In fact, we need terms up to degree 2:

$$\log Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\Lambda) = \frac{1}{\varepsilon_1\varepsilon_2}\Big(\mathcal{F}(\vec{a},\Lambda) + (\varepsilon_1+\varepsilon_2)H(\vec{a},\Lambda) + \varepsilon_1\varepsilon_2 A(\vec{a},\Lambda) + \frac{\varepsilon_1^2+\varepsilon_2^2}{3}B(\vec{a},\Lambda) + \cdots\Big)$$

$$\delta^X_{\mathfrak{z},t}(\underset{\alpha_2=\cdots=0}{\alpha_1 = \overset{H_2(X)}{\alpha z} + \overset{H_0(X)}{p x}}) = \prod_i \delta^{X=\mathbb{C}^2 \text{ around } p_i}_{\mathfrak{z}|_{p_i},t}\left(\Lambda e^{(\alpha z+px)|_{p_i}/4}\right)\Big|_{\varepsilon_1=\varepsilon_2=0} \qquad (x, z : \text{variable})$$

$$= \prod_i Z\left(w(x_i), w(y_i), \frac{t-\mathfrak{z}|_{p_i}}{2}; \Lambda e^{(\alpha z+px)|_{p_i}/4}\right)\Big|_{\varepsilon_1=\varepsilon_2=0}$$

$$= \exp\Big[\sum_i \underbrace{\log Z\left(w(x_i), w(y_i), \frac{t-\mathfrak{z}|_{p_i}}{2}; \Lambda e^{(\alpha z+px)|_{p_i}/4}\right)}_{\parallel}\Big]\Big|_{\varepsilon_1=\varepsilon_2=0}$$

$$\frac{1}{w(x_i)w(y_i)}\Big(\mathcal{F}\left(\frac{t-\mathfrak{z}|_{p_i}}{2}, \Lambda e^{(\alpha z+px)|_{p_i}/4}\right) + (w(x_i)+w(y_i))H\left(\frac{t-\mathfrak{z}|_{p_i}}{2}, \Lambda e^{(\alpha z+px)|_{p_i}/4}\right)$$

$$+ w(x_i)w(y_i)A\left(\frac{t-\mathfrak{z}|_{p_i}}{2}, \Lambda e^{(\alpha z+px)|_{p_i}/4}\right) + \frac{w(x_i)^2+w(y_i)^2}{3}B\left(\frac{t-\mathfrak{z}|_{p_i}}{2}, \Lambda e^{(\alpha z+px)|_{p_i}/4}\right) + \text{higher}\Big)$$

Observation Apply the fixed point formula to $X \supset X^T$ to get

$$\sum \frac{1}{w(x_i)\,w(y_i)} = 0 \qquad \sum \frac{p \mid p_i}{w(x_i)\,w(y_i)} = 1 \qquad \text{etc}$$

$$= \exp\Big[\frac{1}{4}\frac{\partial \mathcal{F}_1}{\partial \log\Lambda} x - \frac{1}{8}\frac{\partial^2 \mathcal{F}_1}{\partial a\, \partial \log \Lambda}\int_X \xi\alpha \cdot z + \frac{1}{32}\frac{\partial^2 \mathcal{F}}{\partial(\log\Lambda)^2}\int_X \alpha^2 \cdot z^2 + \frac{1}{8}\frac{\partial^2 \mathcal{F}}{\partial a^2}\int_X \xi^2$$

$$+ \frac{1}{4}\frac{\partial H}{\partial \log \Lambda}\int_X c_1(X)\alpha \cdot z + \frac{1}{2}\frac{\partial H}{\partial a}\int_X c_1(X)\xi + \chi A + \tau B \Big]$$

(χ = Euler number of X
τ = signature of X)

Since we know $\mathcal{F}$, H, A, B in terms of Seiberg-Witten curve [NY], we can express $\delta^X_{\xi,t}$ in terms of them.
(modular forms, theta functions)

- $H = -\pi\sqrt{-1}\langle \vec{a}, \rho\rangle$ $\qquad \rho$ = half-sum of positive roots
  $= -\frac{\pi\sqrt{-1}}{2}\sum_{\alpha<\beta}(a_\alpha - a_\beta)$

- $\underset{\text{up to const}}{A} = \log\det\left(\frac{\partial u_p}{\partial a_\alpha}\right)$, $\underset{\text{up to const}}{B} = \log \Delta$ descriminant
  (proved only for $r=2$ so far)

This is a main result of [GNY].
And it recovers Göttsche's wall-crossing formula, proved under Kotschick-Morgan conjecture.

Remark. $H = \pi\sqrt{-1}\, a \Rightarrow \partial H / \partial \log \Lambda = 0 \quad , \partial H / \partial a = \pi\sqrt{-1}$
This is compatible with the KM conjecture: $\delta^X_{\mathfrak{z},t}$ is a homotopy invariant up to sign

Otherwise $\delta^X_{\mathfrak{z},t}$ may possibly contains $a(X)$.

2.4. A DETOUR : Geometric Engineering

I postpone to give the proof of Nekrasov's conjecture to the 3rd lecture.

In the remaing time, I will explain some results on:

Question What are the meaning of higher terms in

$$\log Z(\varepsilon_1,\varepsilon_2,\vec{a},\Lambda) = \frac{1}{\varepsilon_1\varepsilon_2}\left(F(\vec{a},\Lambda) + (\varepsilon_1+\varepsilon_2) H(\vec{a},\Lambda) + \varepsilon_1\varepsilon_2 A(\vec{a},\Lambda) + \frac{\varepsilon_1^2+\varepsilon_2^2}{3} B(\vec{a},\Lambda) + \cdots\cdots\right) ?$$

Conjecture All terms can be expressed in terms of the Seiberg-Witten curve.

"Proof". Obvious from the mirror symmetry.

But I will explain other very surprising results suggested from the string theory.

Geometric Engineering (Katz-Klemm-Vafa)

$\Gamma \subset SL_2(\mathbb{C})$ finite subgroup $\longleftrightarrow$ ADE Dynkin diagram

e.g. $\mathbb{Z}_r \Leftrightarrow A_{r-1}$

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \curvearrowleft \Gamma$$
$$\downarrow$$
$$\mathbb{P}^1$$

X_Γ = crepant resolution of X/Γ fiber = minimal resolution of $\mathbb{C}^2/\Gamma$

$\downarrow$

$\mathbb{P}^1$

$$H_2(X_\Gamma, \mathbb{Z}) = \underset{[\mathbb{P}^1]}{\underset{\cup}{H_2(\mathbb{P}^1; \mathbb{Z})}} \oplus \underset{[C_\alpha]}{\underset{\cup}{H_2(\text{fiber}; \mathbb{Z})}}$$

(toric if Γ: type A)

Consider Gromov-Witten invariants for X_Γ (local Calabi-Yau 3-fold).

$$Z^{GW}(\hbar, g_b, g_i)_{i=1,\cdots,r-1} = \exp\left[\sum_{g=0}^{\infty} \hbar^{2g-2} \sum_{d_b=1}^{\infty} g_b^{d_b} \sum_{d_i=0}^{\infty} g_i^{d_i} \int_{[\mathcal{M}_{g,0}(X_\Gamma, \underset{\text{degree}}{d_b, d_i})]^{vir.}} 1\right]$$

$$\boxed{Z^{inst}(\varepsilon_1 = \hbar, \varepsilon_2 = -\hbar, \vec{a}, \Lambda) = \text{a certain limit of } Z^{GW}(\hbar, q_b, q_i)}$$

with a substitution $\begin{cases} \Lambda = q_b \\ q_i = e^{a_{i+1} - a_i} \end{cases}$ (up to const)

Remark, ① If we consider the K-theoretic integration in the instanton side, we do not need to take a limit.

(i.e. The parameter corresponds to the parameter for $K \xrightarrow{gr} H^*$)

② For Γ : type A, $=$ is proved in a mathematically rigorous way by Zhou.

Compute both sides.

a) Recall Z^{inst} was given by the fixed point formula

$M(n,r)^T$ = r-tuples of Young diagrams $(Y_1, \cdots, Y_r)$ $\sum |Y_\alpha| = n$

$\rightsquigarrow$ purely combinatorial expression of Z^{inst}

b) GW side : Use the topological vertex.

$\rightsquigarrow$ Get the same combinatorial expression.

③ perturbative part of the instanton partition function

$=$ GW invariants for $d_b = 0$

up to constant map contribution

Note that the parameters $\varepsilon_1, \varepsilon_2$ $(\in \mathrm{Lie}\, T^2)$ are specialized to the line $\varepsilon_1 + \varepsilon_2 = 0$, $(\hbar = \varepsilon_1 = -\varepsilon_2)$

So a question still remains ....

$\boxed{\text{What is the meaning of } \varepsilon_1 + \varepsilon_2 = c_1^{\mathbb{C}^* \times \mathbb{C}^*}(\mathbb{C}^2) \text{ ?}}$

An explanation was given by Vafa, Gukov ....
Related to Khovanov link homology!

But it is not precise enough, and a further study is required.

—— This is a good point to stop. ——

2.5. A conjectural link to Khovanov homology

Go back to GW invariants for $X_\Gamma = \left(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) / \Gamma \right)^{\sim}$.

How these are computed? (I mentioned the "topological vertex".)

I do not go to the detail.

...... But I want to mention that the invariants are related to Chern-Simons link invariants. (Jones-Witten)

X is a crepant resolution of the conifold.

$$X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \ni ([z_0 : z_1], \zeta_1, \zeta_2)$$

$[z_0 : z_1]$: $\mathbb{P}^1$; ζ_1, ζ_2: fibers

$$\longmapsto (x = z_0\zeta_1, \; y = z_1\zeta_2, \; z = z_0\zeta_2, \; w = z_1\zeta_1) \in \{xy = zw \subset \mathbb{C}^4\}$$

conifold

There is another way to get a smooth manifold from the conifold. $\longrightarrow$ smoothing.

$xy = zw + t$ is smooth if $t \neq 0$ ($t \in \mathbb{C}$: parameter)

diffeomorphic T^*S^3

Large N duality (Gopakumar–Vafa, Ooguri–Vafa, ...)

closed string theory on X_Γ = open string theory on T^*S^3/Γ

GW invariants

open GW invariants with lagrangian $= S^3/\Gamma$

+ Witten SU(N)-Chern-Simons theory on M^3 (real 3-mfd)

= open string theory on T^*M (But make N also as a variable)

This is "computable".

• $\Gamma = \{1\}$ for simplicity ($\rightsquigarrow$ No $\vec{a}$)

Dictionary

	instanton counting	closed GW for X	open GW for T^*S^3	CS for S^3 = colored HOMFLYPT pol.
specialized at $\varepsilon_1+\varepsilon_2=0$	$\varepsilon_1=-\varepsilon_2$, Λ	$\hbar$, g_0 (genus) (degree)	$\hbar$, t (genus), (# of holes)	N, k (rank) (level)

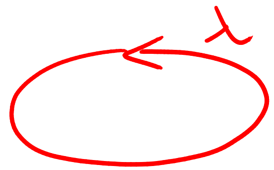
$\Downarrow$ 1 additional variable

$\varepsilon_1+\varepsilon_2$	?	?	Poincaré polynomial of triply graded link homology group

(Euler char. of link homology = colored HOMFLYPT)

Unfortunately this is very speculative, as triply graded link homology is defined only for the vector representations (and probably exterior powers)

But still, we can make nontrivial checks.

Consider the unknot  in S^3

decorated by a representation λ of $SU(N)$

(Young diagram)

Chern-Simons Jones Witten inv. $=$ "quantum" dimension of λ

(deformation of the actual dim.)

In open or closed GW invariants, it is expected that it corresponds to a Lagrangian subvariety in X or T^*S^3

$\cup$ conormal b'dle of the link + information of λ

In the instanton counting, it is expected that it corresponds to a natural vector b'dle over the resolution of the moduli spaces :

$U(1)$-case : resolution = Hilbert scheme of points on $\mathbb{C}^2$

$$= \{ I \subset \mathbb{C}[x,y] \mid \text{ideals}, \dim \mathbb{C}[x,y]/I = n \}$$

$$M(1,n) \longrightarrow M_0(1,n) = S^n \mathbb{R}^4$$

Let E be a vector b'dle over $M(1,n)$, whose fiber at I $= \mathbb{C}[x,y]/I$ (tautological line b'dle)

$$\sum_{n=0}^{\infty} \sum_k (-1)^k \mathrm{ch}\, H^k_{T^2}(M(1,n), S^\lambda E)\, \Lambda^{2n}$$

S^λ : Schur functor
e.g $\Lambda^p E$, $S^p E$ etc

(K-theoretic correlation function for $U(1)$-gauge theory)

This is essentially equals to the Poincaré polynomial of Khovanov homology of the unknot.
(not yet defined beyond $\Lambda^p E$)