2.5. A conjectural link to Khovanov homology

RECALL

$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$ instanton patition function

$= $ GW invariants for $X_\Gamma = \left(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) / \Gamma \right)^{\sim}$,

when we specialise $\varepsilon_1 = \hbar$, $\varepsilon_2 = -\hbar$.

How RHS was computed? (I mentioned the "topological vertex".)

I do not go to the detail.

...... But I want to mention that it is based on to a relation to Chern-Simons link invariants (Jones-Witten) by large N duality.

Quick Review of Large N duality (Gopakumar-Vafa, Ooguri-Vafa, ...)

Recall X is a crepant resolution of the conifold.

$$☺\quad X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \ni ([\underset{\underset{\mathbb{P}^1}{\uparrow}}{z_0 : z_1}], \zeta_1, \zeta_2)$$

fibers

conifold

$$\longmapsto (x = z_0\zeta_1, \; y = z_1\zeta_2, \; z = z_0\zeta_2, \; w = z_1\zeta_1) \in \{xy = zw \subset \mathbb{C}^4\}$$

There is another way to get a smooth manifold from the conifold. $\longrightarrow$ smoothing.

$xy = zw + t$ is smooth if $t \neq 0$

($t \in \mathbb{C}$: parameter)

diffeomorphic

T^*S^3

Then the duality says

open-closed string theory on X_Γ = open string theory on T^*S^3/Γ

GW invariants

open GW invariants with lagrangian $= S^3/\Gamma$

+ Witten SU(N)-Chern-Simons theory on M^3 (real 3-mfd)

= open string theory on T^*M (But make N also as a variable)

This is "computable".

- $\Gamma=\{1\}$ for simplicity (⇝ No $\vec{a}$)

Dictionary

	instanton counting	closed GW for X	open GW for T^*S^3	CS for S^3 = colored HOMFLYPT pol.
specialised at $\varepsilon_1+\varepsilon_2=0$	$\varepsilon_1=-\varepsilon_2$, Λ	$\hbar$, g_b (genus) (degree)	$\hbar$, t (genus) (# of holes)	N, k (rank) (level)
	$\varepsilon_1+\varepsilon_2$	?	?	Poincaré polynomial of triply graded link homology group

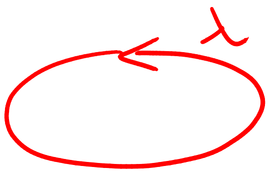
$\Downarrow$ 1 additional variable

(Euler char. of link homology = colored HOMFLYPT)

Unfortunately this is very speculative, as triply graded link homology is defined only for the vector representations (and, probably exterior powers)

But still, we can make nontrivial checks.

Consider the unknot



in S^3

decorated by a representation λ of $SU(N)$
(Young diagram)

Chern-Simons
Jones Witten inv. = "quantum" dimension of λ
(deformation of the actural dim.)

In open or closed GW invariants, it is expected that
it corresponds to a lagrangian subvariety
in X or T^*S^3
$\cup$ conormal b'dle of the link
+ information of λ

In the instanton counting, it is expected that it corresponds to a natural vector b'dle over the resolution of the moduli spaces :

$U(1)$-case: $M(1,n)$ = Hilbert scheme of points on $\mathbb{C}^2$
$= \{ I \subset \mathbb{C}[x,y] \mid$ ideals, $\dim \mathbb{C}[x,y]/I = n \}$

Let E be a vector b'dle over $M(1,n)$, whose fiber at I
$= \mathbb{C}[x,y]/I$ (tautological line b'dle)

$$\sum_{n=0}^{\infty} \sum_{k} (-1)^k \mathrm{ch}\, H^k_{T^2}(M(1,n), S^\lambda E)\, \Lambda^{2n}$$

S^λ : Schur functor
e.g $\Lambda^p E$, $S^p E$ etc

(K-theoretic correlation function for $U(1)$-gauge theory)

This is essentially equals to the Poincaré polynomial of Khovanov homology of the unknot.
(not yet defined beyond $\Lambda^p E$)

PART III. Blow-up formula via Wall-crossing

3.1. Proof of Nekrasov Conjecture : Strategy

N-Yoshioka
Nekrasov-Okounkov
Braverman-Etingof

All proofs are very different.

Also a new approach is recently found by Klemm-Sułkowski via matrix integrals.

Our proof is based on an old idea due to Fintushel-Stern, Göttsche.....

Compare Donaldson invariants for X and its one-point blow-up $\hat{X}$.

$M(n,r)$ = framed moduli space of torsion-free sheaves on $\mathbb{P}^2 = \mathbb{C}^2 \cup l_\infty$

$\hat{M}(n,k,r)$ = ——— " ——— $\hat{\mathbb{P}}^2 = \hat{\mathbb{C}}^2 \cup l_\infty$

↑ blow-up at 0

We compare instanton partition functions on $\mathbb{C}^2$ & $\hat{\mathbb{C}}^2$.

$$Z^{inst}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\alpha};\Lambda)=\sum_{n=0}^{\infty}\Lambda^{2nr}\int_{M(n,r)}\exp\Big(\sum_{p=1}^{\infty}(-1)^p\,ch_{p+1}(\mathcal{E})/_{[\mathbb{C}^2]}\cdot\alpha_p\Big)$$

$$\vec{\alpha}=(\alpha_1,\alpha_2,\cdots\cdots)$$

$$\hat{Z}^{inst}_{c_1=k}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\alpha},\vec{\tau};\Lambda)$$
$$=\sum_{n=0}^{\infty}\Lambda^{2nr}\int_{\hat{M}(n,k,r)}\exp\Big(\sum_{p=1}^{\infty}(-1)^p\,ch_{p+1}(\mathcal{E})/_{(\alpha_p[\hat{\mathbb{C}}^2]+\tau_p[C])}\Big)$$

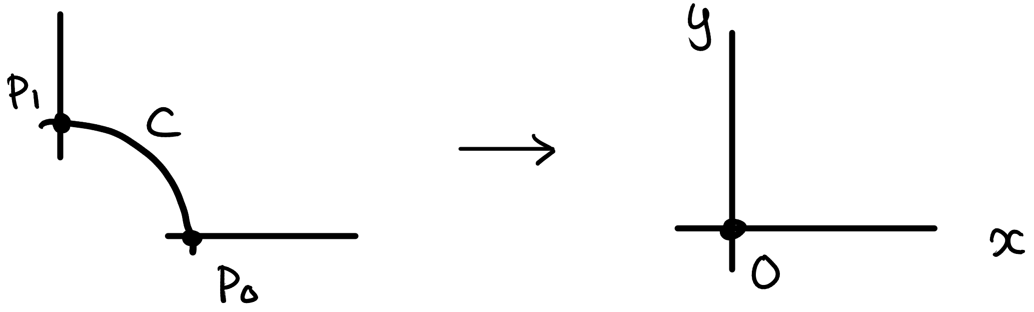
where C is the exceptional locus of the blow-up.

By the fixed point formula

$\hat{Z}$ can be expressed by Z.

This is the same technique used to write $\tilde{\delta}^{\overset{\text{toric}}{X}}_{3,t}$ in terms of $\tilde{\delta}^{\mathbb{C}^2}_{3,t}$.

$T^2 \curvearrowright \hat{\mathbb{C}}^2$ Two fixed pts



The formula is simplified if one include perturbative parts.

$$\boxed{\begin{aligned}&\hat{Z}_{c_1=k}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\alpha},\vec{\tau};\Lambda)\\ &= \sum_{\substack{\vec{k}\in\mathbb{Z}^r\\ :\sum k_\alpha = k}} Z(\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+\varepsilon_1\vec{k},\vec{\alpha}+\varepsilon_1\vec{\tau};\Lambda)\, Z(\varepsilon_1-\varepsilon_2,\varepsilon_2,\vec{a}+\varepsilon_2\vec{k},\vec{\alpha}+\varepsilon_2\vec{k};\Lambda)\end{aligned}}$$

- Two Z ···· contribution from p_0 and p_1.

$$\begin{cases}(\varepsilon_1,\varepsilon_2-\varepsilon_1) & T^2\text{-weights of } T_{p_0}\hat{\mathbb{C}}^2\\ (\varepsilon_1-\varepsilon_2,\varepsilon_2) & \text{———— '' ————}\ T_{p_1}\hat{\mathbb{C}}^2\end{cases}$$

- $\sum_{\vec{k}\in\mathbb{Z}^r}$ comes from sum of line bundles

$$\mathcal{O}(k_1C)\oplus\mathcal{O}(k_2C)\oplus\cdots\oplus\mathcal{O}(k_rC)$$

$$\hat{M}(r,k,n)^{\tilde{T}} = \{(E,\Phi) \mid E = \mathcal{I}_1(k_1C)\oplus\cdots\oplus\mathcal{I}_r(k_rC) \quad \}$$

$\mathcal{I}_\alpha \subset \mathcal{O}_{\hat{\mathbb{C}}^2}$ ideal sheaf

$\mathcal{O}_{\hat{\mathbb{C}}^2}/\mathcal{I}_\alpha$ supported at $p_0 \sqcup p_1$

& a monomial ideal

w.r.t. toric coord.

On the other hand, we can compare $M(r,n)$ and $\hat{M}(r,k,n)$ more directly.

$\exists$ projective morphism $\hat{\pi} : \hat{M}(r,k,n) \longrightarrow M_0(r,n')$

$$\begin{array}{ccc} \hat{M}(r,k,n) & & \\ \rotatebox{90}{$\in$} & & \\ (E,\Phi) & \longmapsto & ((p_*E)^{\vee\vee}, \Phi) + \text{points with mult.} \end{array}$$

Proposition $\displaystyle\int_{\hat{M}(r,0,n)} (ch_2(\mathcal{E})/[C])^d = 0$ for $1 \leqq d \leqq 2r-1$

(proof) $\hat{M}(r,0,n) \xrightarrow{\hat{\pi}} M_0(r,n) = M_0^{reg}(r,n) \sqcup M_0^{reg}(r,n-1) \times \mathbb{C}^2 \sqcup \cdots$

$\odot$ $ch_2(\mathcal{E})/[C] = 0$ on the complement of $\pi^{-1}(M_0(r,n-1)\times\{0\})$,

as otherwise $\mathcal{E}|_C$ is trivial. $\uparrow$ codim $= 2r$ in $M_0(r,n)$

//

$\Rightarrow$ This leads to a functorial equation on Z, which determines coefficients of Z in Λ^{2nr} recursively in $n = c_2$.

$\Rightarrow$ The equation remains to hold at $\varepsilon_1 = \varepsilon_2 = 0$.
And the same equation hold on the Seiberg-Witten prepotential.

$\Rightarrow$ $\varepsilon_1\varepsilon_2 \log Z \Big|_{\varepsilon_1=\varepsilon_2=0}$ = SW prepotential !

HOWEVER

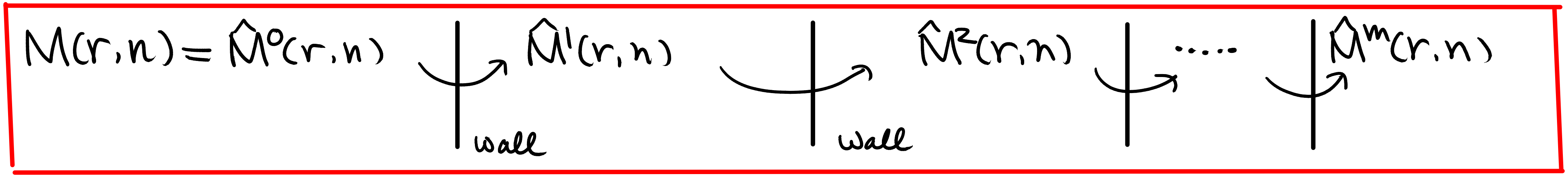
This approach does NOT work when we include $\exp(ch_{p+1}(\mathcal{E})/[\hat{\mathbb{C}}^2]\,\alpha_p)$.

$\Rightarrow$ Need a closer look to relation between $\widehat{M}(r,k,n)$ & $M(r,n)$.

3.2 Perverse coherent sheaves on blow-up

IDEA

Using the notion of perverse coherent sheaves due to Bridgeland, we connect $\hat{M}(r,k,n)$ and $M(r,n)$ via wall-crossing.



s.t. $\hat{M}^m(r,n) = \hat{M}(r,0,n)$ for $m \gg$ constant depending on r,n.

$\hat{M}^m(r,n)$: framed moduli space of m-stable perverse coherent sheaves on $\hat{\mathbb{P}}^2 = \hat{\mathbb{C}}^2 \cup \ell_\infty$.

Def. (E, Φ) : m-stable perverse coherent

$\overset{\text{def}}{\Longleftrightarrow}$
- E : coherent sheaf on $\hat{\mathbb{P}}^2$, torsion free outside C
- $\Phi : E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$
- $\boxed{\mathrm{Hom}(E, \mathcal{O}_C(-m-1)) = 0, \quad \mathrm{Hom}(\mathcal{O}_C(-m), E) = 0}$

Remark. E has torsion in general.

By applying Mochizuki's theory

$$\int_{\hat{M}^{m}(r,n)} \Phi - \int_{\hat{M}^{m+1}(r,n)} \Phi = \sum_{n'<n} \int_{\hat{M}^{m}(r,n')} \Phi \times \underbrace{\int_{\text{Grassmannian}}}_{\text{a product}} \boxed{\sim\sim\sim}$$

(an exceptional fixed point

$$E = E' \oplus \mathcal{O}_C(-m-1)^{\oplus p}$$

The previous vanishing is a special case.

Review of the theory of perverse coherent sheaves

Bridgeland, Van den Bergh

- $p : Y \to X$ projective morphism between quasi-projective varieties
  s.t. · $\dim p^{-1}(x) \leqq 1 \quad \forall x \in X$
  · $\mathbb{R}p_* \mathcal{O}_Y = \mathcal{O}_X$

e.g. ∘ $p : \hat{X} \to X$ blow-up of a surface at a nonsingular point
∘ $p : \mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \longrightarrow \{xy = zw \subset \mathbb{C}^4\}$
crepant resolution of conifold

Def. $\mathcal{C} := \{ K \in \mathrm{Coh}(Y) \mid \mathbb{R}p_* K = 0 \}$
$\mathcal{T} := \{ E \in \mathrm{Coh}(Y) \mid \mathbb{R}^1 p_* E = 0 \,,\ \mathrm{Hom}(E, K) = 0 \quad \forall K \in \mathcal{C} \}$
$\mathcal{F} := \{ E \in \mathrm{Coh}(Y) \mid p_* E = 0 \}$

Prop. $(\mathcal{T}, \mathcal{F})$ is a torsion pair of $\mathrm{Coh}\, Y$
i.e. ∘ $\mathrm{Hom}(T, F) = 0 \quad \forall T \in \mathcal{T} \quad \forall F \in \mathcal{F}$
∘ $\forall E \quad \exists T, F$ s.t. $0 \to T \to E \to F \to 0$ exact

Cor. $\mathrm{Per}(Y/X) := \{ E \in D^b(Y) \mid H^i(E) = 0 \quad i \neq 0, -1$
$H^{-1}(E) \in \mathcal{F} \,,\ H^0(E) \in \mathcal{T} \}$
is the heart of a t-structure on $D^b(Y)$.
In particular, it is an abelian category.

Bridgeland constructed the moduli space of perverse ideal sheaves,
i.e.,

$$I \subset \underset{\substack{\uparrow \\ \mathrm{Per}(Y/X)}}{\mathcal{O}_Y} \text{ in } \mathrm{Per}(Y/X)$$

Consider an ideal sheaf of colength 1 for $p : \hat{X} \to X$
blowup of surface

a) $x \in \hat{X} \setminus C \quad \Rightarrow \quad I_x \in \mathrm{Per}(\hat{X}/X)$

b) $x \in C \quad \Rightarrow \quad I_x \notin \mathrm{Per}(\hat{X}/X)$

$$0 \to \mathcal{O}(-C) \to I_x \xrightarrow{\ ☹\ } \underset{\substack{\uparrow \\ \mathcal{E}}}{\mathcal{O}_C(-1)} \to 0 \qquad \mathrm{Hom}(I_x, \mathcal{O}_C(-1)) \neq 0$$

To get an object in $\mathrm{Per}(\hat{X}/X)$, ↔ Exchange

$$0 \to \mathcal{O}_C(-1) \to \underset{\substack{\uparrow \\ \mathrm{Per}(\hat{X}/X)}}{E} \to \mathcal{O}(-C) \to 0$$

Such an E is unique up to isom. as $\dim \mathrm{Ext}^1(\mathcal{O}(-C), \mathcal{O}_C(-1)) = 1$.

$\therefore$ moduli $= X$ instead of $\hat{X}$!

$$X \underset{\text{wall}}{\overset{\leftarrow}{\Big|}} \hat{X}$$

Remark Bridgeland $Y \to X$: small crepant resolution of 3-fold
$\Rightarrow$ moduli of perverse coherent ideal sheaves of colength $= 1$ $\cong Y^+$: flop of f
$\Rightarrow$ $D^b(\mathrm{Coh}\, Y) \cong D^b(\mathrm{Coh}\, Y^+)$ by Fourier-Mukai

Def (1) $X =$ affine $P \in \mathrm{Per}(Y/X)$ is a projective generator if it is a projective object in $\mathrm{Per}(Y/X)$ and $\mathrm{Hom}_{\mathrm{Per}(Y/X)}(P, E) = 0 \Rightarrow E = 0$

e.g. $Y = \hat{\mathbb{C}}^2 \to X = \mathbb{C}^2$ $P = \mathcal{O}_Y \oplus \mathcal{O}_Y(-C)$

(2) In general, $P \in \mathrm{Per}(Y/X)$ is a local projective generator if $X = \overset{\exists}{\bigcup} U_i$ open covering s.t. $P|_{U_i}$: projective generator

THEOREM (Van den Bergh)

$$\begin{tikzcd}
D^b(\mathrm{Coh}\, Y) \arrow[r, shift left, "\mathbb{R}p_* \mathbb{R}\mathcal{H}om(P, \bullet)"] \arrow[r, phantom, "\cong"] & D^b(\mathrm{mod}\text{-}(\overset{/\!/ \mathcal{A}}{\mathbb{R}p_* \mathcal{E}nd(P)})) \arrow[l, shift left, "\bullet \otimes^L_{\mathcal{A}} P"] \\
\mathrm{Per}(Y/X) \arrow[u, phantom, "\cup" sloped] \arrow[r, phantom, "\cong"] & \mathrm{mod}\, \mathcal{A} \arrow[u, phantom, "\cup" sloped]
\end{tikzcd}$$

We can construct moduli spaces of "stable" perverse coherent sheaves as moduli spaces of objects in mod $\mathcal{A}$ via Simpson's method.
(By a technical reason, we used a slight modification.)

THEOREM (N+Yoshioka)

Consider the case $p : \hat{X} \to X$ blowup of a surface ,

& • support of $\mathbb{R}p_* \mathbb{R}\mathrm{Hom}(P, E)$: 2 dim'l

• coprime condition holds on X

(stability ⟺ semistability)

Then $E \in \mathrm{Per}(\hat{X}/X)$ is stable (⟺ semistable)

⟺ • $E \in \mathrm{Coh}\,\hat{X}$

• $p_* E$ is torsion free and stable

(In the framed case) ⟺ 0-stable

i.e. • torsion free outside C

• $\mathrm{Hom}(E, \mathcal{O}_C(-1)) = 0$, $\mathrm{Hom}(\mathcal{O}_C, E) = 0$

∴ moduli space of stable perverse coherent sheaves on X $\xrightarrow{p_*}$ moduli space of stable sheaves on X

m-stable $\overset{\text{def.}}{\iff}$ $E(mC)$ is 0-stable

Rem. $\mathrm{Per}(\hat{X}/X)$ is not preserved by $\otimes\, \mathcal{O}(C)$.

We analyse what happens

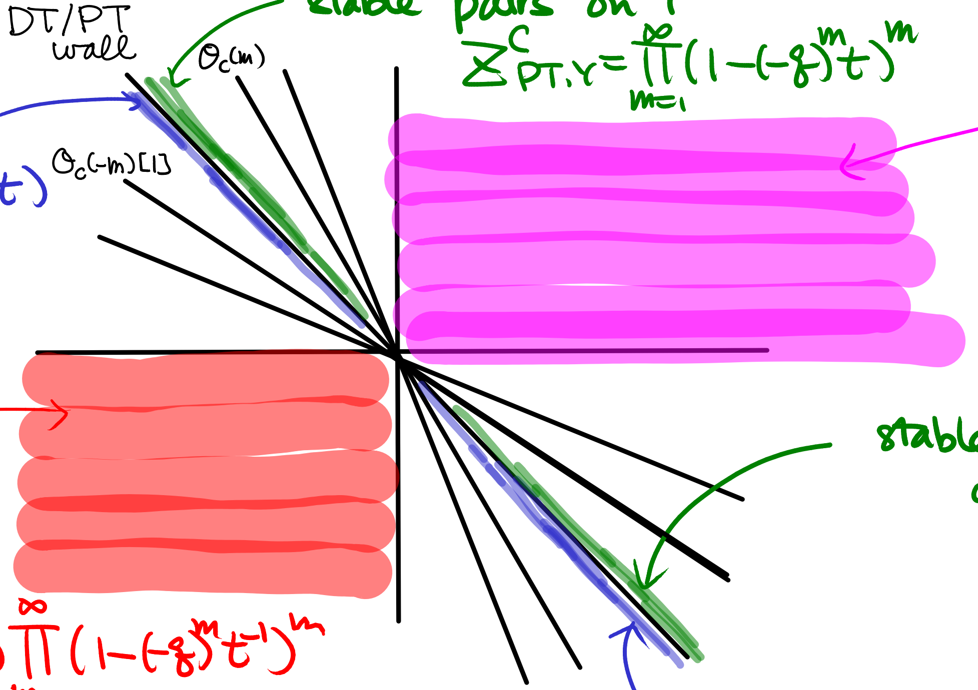
E : m-stable but not $(m\pm1)$-stable

$\Rightarrow$ wall-crossing

Technically we use a quiver description of the moduli space to apply the theory of the master space

(just scheme does not work.)

Remark We have a similar wall-crossing for $p: \underset{\overset{\|}{\mathcal{O}(-1)\oplus\mathcal{O}(-1)}}{Y} \longrightarrow X$ resolved conifold.
(w/ K. NAGAO)



ideal sheaves on Y
$$Z_{DT,Y} = \left(\prod_{m=1}^{\infty}(1-(-q)^m)^{-m}\right)^2 \times \prod_{m=1}^{\infty}(1-(-q)^m t)$$

stable pairs on Y
$$Z^{C}_{PT,Y} = \prod_{m=1}^{\infty}(1-(-q)^m t)^m$$

$Z_{triv.} \equiv 1$

only $\mathcal{O}_Y$ is stable

$Per(Y/X)$
$=$ Szendroi's noncomm. DT
$$Z_{NCDT} = \left(\prod_{m=1}^{\infty}(1-(-q)^m)^{-m}\right)^2 \times \prod_{m=1}^{\infty}(1-(-q)^m t)\prod_{m=1}^{\infty}(1-(-q)^m t^{-1})^m$$

stable pairs on Y^+: flop
$$Z_{PT,Y^+} = \prod_{m=1}^{\infty}(1-(-q)^m t^{-1})^m$$

Ideal sheaves on Y^+
$$Z_{DT,Y^+} = \left(\prod_{m=1}^{\infty}(1-(-q)^m)^{-m}\right) \times \prod_{m=1}^{\infty}(1-(-q)^m t^{-1})$$

$\Longrightarrow$ **Example of Wall-crossing formula of DT inv.**

Since $\mathcal{O}_C(-m)$ has no self-extensions, a direct calculation is very simple. It is even simpler than the DT/stable pair -wall explained by Thomas.