2.5. A conjectural link to Khovanov homology.

RECALL

\[ Z(\varepsilon_1, \varepsilon_2, \Delta) \text{ instanton partition function} \]

\[ = GW \text{ invariants for } X_P = (\mathcal{O}_P^1 \oplus \mathcal{O}_P^1) / \mathbb{P}^{1} \mathbb{P}^{1} \]

When we specialize \( \varepsilon_1 = \hbar \), \( \varepsilon_2 = -\hbar \).

How RHS was computed? (I mentioned the "topological vertex").

I do not go to the detail.

...... But I want to mention that it is based on to a relation to Chem-Simons link invariants (Jones-Witten) by large N duality.
Quick Review of Large N duality (Gopakumar-Vafa, Ooguri-Vafa, \ldots)

Recall $X$ is a crepant resolution of the conifold.

$$X = \mathbb{C}^* \times \mathbb{C}^* \cong \{ (z_0 : z_1, z_1, z_2) \}$$

\begin{array}{c}
\uparrow \\
\mathbb{P}^1 \text{ fibers}
\end{array}

\begin{array}{c}
\xrightarrow{\text{conifold}} \\
C^4
\end{array}

\rightarrow (x = z_0 z_1, y = z_1 z_2, z = z_0 z_2, w = z_1 z_1) \in \{ x y = z w \subset C^4 \}

There is another way to get a smooth manifold from the conifold. \xrightarrow{\text{smoothing}}

$\{ x y = z w + t \}$ is smooth if $t \neq 0$

$t \in \mathbb{C} : \text{parameter}$

$T^* S^3$

diffeomorphic

Then the duality says open-closed string theory on $X^\dagger = \text{open string theory on } T^* S^3 / \Gamma$

GW invariants

open GW invariants with lagrangian = $S^3 / \Gamma$
Written SU(N)-Chern-Simons theory on $M^3$ (real 3-mfld) = open string theory on $T^*M$ (But make $N$ also as a variable)

This is "computable".

$\Gamma = n$ for simplicity ($\rightarrow$ No $\tilde{a}$)

**Dictionary**

<table>
<thead>
<tr>
<th>Instanton counting</th>
<th>Closed GW for $X$</th>
<th>Open GW for $T^*S^3$</th>
<th>CS for $S^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1 = -\xi_2, \Delta$</td>
<td>$h$, $g_0$ (genus) (degree)</td>
<td>$h$, $t$ (genus) (# of holes)</td>
<td>$N$, $r$ (rank) (level)</td>
</tr>
</tbody>
</table>

specialized at $\xi_1 + \xi_2 = 0$

\[ \downarrow \text{1 additional variable} \]

\[ \xi_1 + \xi_2 \]

$?$ $?$ $?$

\[ \text{(Euler char. of link homology = colored HOMFLYPT)} \]
Unfortunately this is very speculative, as triply graded link homology is defined only for the vector representations (and external powers) probably.

But still, we can make nontrivial checks.

Consider the unknot in $S^3$

decorated by a representation $\lambda$ of $SU(N)$

\begin{equation*}
\text{Chern-Simons} = \text{"quantum" dimension of } \lambda \\
\text{Jones-Witten inv.} = \text{deformation of the actual dim.}
\end{equation*}

In open or closed GW invariants, it is expected that if it corresponds to a Lagrangian subvariety in $X \times \mathbb{T}^4 S^3$

\begin{equation*}
\cup \text{conormal bundle of the link} + \text{information of } \lambda
\end{equation*}
In the instanton counting, it is expected that it corresponds to a natural vector bundle over the resolution of the moduli spaces:

\[ \mathcal{U}(1) \text{-case: } M(1,n) = \text{Hilbert scheme of points on } \mathbb{C}^2 \]
\[ = \{ I \subset \mathbb{C}[x,y] \mid \text{ideals, } \dim \mathbb{C}[x,y] / I = n \} \]

Let \( E \) be a vector bundle over \( M(1,n) \), whose fiber at \( I \)
\[ = \mathbb{C}[x,y] / I \]
(tautological line bundle)

\[ \sum_{n=0}^{\infty} \sum_{k} (-1)^k \chi_{T^2} (M(1,n), S^k E) \Delta^{2n} \]
\[ S^k : \text{Schur function, e.g. } \Lambda^k E, S^k E \text{ etc.} \]

\[ \text{(K-theoretic correlation function for } \mathcal{U}(1)\text{-gauge theory)} \]

This is essentially equal to the Poincaré polynomial of Khovanov homology of the unknot, (not yet defined beyond \( \Lambda^0 E \))
PART III. Blow-up formula via Wall-crossing

3.1. Proof of Nekrasov Conjecture: Strategy

All proofs are very different.

Also a new approach is recently found by Klemm-Sulkowski via matrix integrals.

Our proof is based on an old idea due to Fintushel-Stern, Göttsche, ....

Compare Donaldson invariants for $X$ and its one-point blow-up $\hat{X}$.

$M(n,r) =$ framed moduli space of torsion-free sheaves on $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{C}^2$

$\hat{M}(n,k,r) =$ " " $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{C}^2$

We compare instanton partition functions on $\mathbb{C}^2$ & $\mathbb{C}^2$. 
\[ \Xi^\text{inst}_e (e_1, e_2, \vec{x}, \vec{a}; \lambda) = \sum_{n=0}^{\infty} \Delta^{2nr} \int_{M(G, r)} \exp \left( \sum_{p=1}^{\infty} (-1)^p \alpha_s \rho_{e_1} (E) / [C^p] \cdot \alpha_p \right) \]

\[ \bar{\alpha} = (\alpha_1, \alpha_2, \ldots) \]

where \( C \) is the exceptional locus of the blow-up.

By the fixed point formula

\[ \Xi \]

can be expressed by \( \Xi \).

This is the same technique used to write \( \Xi_{X_{3,t}}^{\text{toric}} \) in terms of \( \Xi_{C^2} \).

\[ T^2 \sim C^2 \] Two fixed pts

\[ P_1 \]

\[ P_0 \]

\[ y \]

\[ x \]
The formula is simplified if one include perturbative parts.

\[
\bigwedge_{C_i = \emptyset} (e_1, e_2, \tilde{a}', \tilde{a}, \tilde{c}', \lambda)
\]
\[
= \sum_{\mathbf{\epsilon} \in \mathbb{Z}^r : \sum \mathbf{\epsilon}_i = \mathbf{r}} \bigwedge (e_1, e_2 - e_1, \tilde{a}' + \mathbf{\epsilon}, \tilde{a} + \mathbf{\epsilon}, \tilde{c}' + \mathbf{\epsilon}_1 \lambda ; \lambda) \bigwedge (e_1 - e_2, e_2, \tilde{a}' + \mathbf{\epsilon}_2 \tilde{a}', \tilde{a} + \mathbf{\epsilon}_2 \tilde{c}' ; \lambda)
\]

- Two \( \bigwedge \) .... contribution from \( p_0 \) and \( p_1 \).

\[
\bigwedge (e_1 - e_2, e_2) \quad T^2 \text{weights of } \hat{T}_{p_0} \hat{C}^2
\]
\[
\bigwedge (e_1, e_2 - e_1) \quad \text{--- } \hat{T}_{p_1} \hat{C}^2
\]

- \( \sum \) comes from sum of line bundles

\( \mathcal{O}(c_1 C) \oplus \mathcal{O}(c_2 C) \oplus \ldots \oplus \mathcal{O}(c_r C) \)

\( \hat{M}(r, \mathbb{R}, n) \)

\[
\hat{M}(r, \mathbb{R}, n) = \{ (E, \mathbf{z}) \mid E = \mathcal{O}(c_1 C) \oplus \ldots \oplus \mathcal{O}(c_r C) \}
\]
\[
J_{aC} \subset \mathcal{O}_{\hat{C}^2} \quad \text{ideal sheaf}
\]
\[
\mathcal{O}_{\hat{C}^2} / J_{aC} \text{ supported at } p_0 \cup p_1
\]
\[ \mathbf{z} \text{ a monomial ideal w.r.t. tonic coord.} \]
On the other hand, we can compare $\tilde{M}(r,n)$ and $\hat{M}(r,\mathbb{P},n)$ more directly.

\[ \exists \text{ projective morphism} \quad \pi : \hat{M}(r,\mathbb{P},n) \rightarrow M_0(r,n) \]

\[ (E, \varpi) \longmapsto ((p_* E)^\mathbb{P}, \varpi) + \text{ points with mult.} \]

**Proposition**

\[ \int_{\tilde{M}(r,0,n)} (\chi(E)/[C])^d = 0 \quad \text{for} \quad 1 \leq d \leq 2r-1 \]

**(proof)**

\[ \hat{M}(r,0,n) \xrightarrow{\pi} M_0(r,n) = M_{0}^{\text{reg}}(r,n) \sqcup M_{0}^{\text{reg}}(r,n-1) \times \mathbb{C}^2 \sqcup \cdots \]

\[ (\chi(E)/[C] = 0 \quad \text{on the complement of} \quad \pi^{-1}(M_0(r,n-1) \times \mathbb{P}^2)) \]

as otherwise $E|_C$ is trivial.\[ \text{codim} = 2r \quad \text{in} \quad M_0(r,n) \]
This leads to a functional equation on $\mathcal{Z}$, which determines coefficients of $\mathcal{Z}$ in $\Delta^{2\text{nd}}$ recursively in $n=c_2$.

The equation remains to hold at $\zeta_1 = \zeta_2 = 0$.

And the same equation hold on the Seiberg-Witten prepotential.

$$\gamma \log \mathcal{Z} \bigg|_{\zeta_1 = \zeta_2 = 0} = \text{SW prepotential} \quad \Box$$

**HOWEVER**

This approach does **NOT** work when we include $\exp(\Delta_{\mu}(\zeta)[\xi] \omega)$. 

Need a closer look to relation between $\hat{M}(r,\xi,n)$ & $M(r,\xi,n)$. 
3.2 Perverse coherent sheaves on blow-up

IDEA

Using the notion of perverse coherent sheaves due to Bridgeland, we connect \( \hat{M}(r,n) \) and \( M(r,n) \) via wall-crossing.

\[
\begin{align*}
M(r,n) &= \hat{M}^0(r,n) \quad \xrightarrow{\text{wall}} \quad \hat{M}^1(r,n) \quad \xrightarrow{\text{wall}} \quad \hat{M}^2(r,n) \quad \ldots \quad \hat{M}^m(r,n)
\end{align*}
\]

s.t. \( \hat{M}^m(r,n) = \hat{M}(r,0,n) \) for \( m \Rightarrow \text{constant depending on } r,n. \)

\( \hat{M}^m(r,n) \): framed moduli space of \( m \)-stable perverse coherent sheaves on \( \mathbb{P}^2 = \mathbb{P}^2 \cup \text{loc}. \)

\[\text{Def. } (E, \Phi) \text{: } m \text{-stable perverse coherent} \]

\[\begin{align*}
\bullet & \ E: \text{coherent sheaf on } \mathbb{P}^2, \text{torsion free outside } C \\
\bullet & \ \Phi: E|_{\text{loc}} \cong O_{\text{loc}} \\
\bullet & \ \text{Hom}(E, O_C(-m-1)) = 0, \text{ Hom}(O_C(-m), E) = 0
\end{align*}\]

Remark. \( E \) has torsion in general.
By applying Modinuki's theory

$S_{\operatorname{Mor}(r,n)} \to S_{\operatorname{Mor}^1(r,n)} = \sum_{n' < n} S_{\operatorname{Mor}(r,n')} \otimes S_{\text{Grassmannian}}$  (a product)

( an exceptional fixed point

$E = E \otimes \Omega_{\mathbb{C}(-m-1)}$)

The previous vanishing is a special case.
Review of the theory of perverse coherent sheaves

Bridgeland, Van den Bergh

- \( p : Y \to X \) projective morphism between quasi-projective varieties
- \( \dim p^{-1}(x) \leq 1 \quad \forall x \in X \)
- \( R^i p_* \mathcal{O}_Y = 0 \)
- \( p : T_{\mathbb{R}}(\mathcal{O}(1) \oplus \mathcal{O}(-1)) \to \{ xy = zw \subset \mathbb{C}^4 \} \)
- crepant resolution of conifold

\textbf{Def.} \( C := \{ K \in \operatorname{Coh}(Y) \mid R^i p_* K = 0 \} \)
\( \mathfrak{T} := \{ E \in \operatorname{Coh}(Y) \mid R^1 p_* E = 0, \operatorname{Hom}(E, K) = 0 \quad \forall K \in C \} \)
\( \mathfrak{Z} := \{ E \in \operatorname{Coh}(Y) \mid p_* E = 0 \} \)

\textbf{Prop.} \((\mathfrak{T}, \mathfrak{Z})\) is a torsion pair of \( \operatorname{Coh}(Y) \)
\( \text{i.e.} \quad \operatorname{Hom}(T, F) = 0 \quad \forall T \in \mathfrak{T}, \forall F \in \mathfrak{Z} \)
\( \forall E \quad \exists T, F \quad \text{s.t.} \quad 0 \to T \to E \to F \to 0 \quad \text{exact} \)

\textbf{Cor.} \( \operatorname{Per}(Y/X) := \{ E \in D^b(Y) \mid H^i(E) = 0 \quad i \neq 0, -1 \} \)
\( H^{-1}(E) \in \mathfrak{T}, \quad H^0(E) \in \mathfrak{Z} \)

is the heart of a \( t \)-structure on \( D^b(Y) \).
In particular, it is an abelian category.
Bridgeland constructed the moduli space of perverse ideal sheaves, i.e.,
\[ J \subset \mathcal{O}_X \text{ in } \text{Per}(Y/X) \]
\[ \text{Per}(Y/X) \]

Consider an ideal sheaf of codimension 1 for \( p: \hat{X} \to X \) blowup of surface

a) \( x \in \hat{X} \setminus C \implies J_x \in \text{Per}(\hat{X}/X) \)
b) \( x \in C \implies J_x \notin \text{Per}(\hat{X}/X) \)

\[ 0 \to \mathcal{O}(-C) \to J_x \to \mathcal{O}_C(-1) \to 0 \]
\[ \text{Hom}(J_x, \mathcal{O}_C(-1)) = 0 \]

To get an object in \( \text{Per}(\hat{X}/X) \)

\[ 0 \to \mathcal{O}_C(1) \to E \to \mathcal{O}(-C) \to 0 \]
\[ \text{dim Ext}^1(\mathcal{O}(-C), \mathcal{O}_C(1)) = 1. \]

Such an \( E \) is unique up to isom. as

\[ \text{moduli } = X \text{ instead of } \hat{X}! \]

\[ X \xrightarrow{\text{well}} \hat{X} \]
**Remark**  Bridgeland $Y \to X$: small crepant resolution & 3-fold
  $\Rightarrow$ moduli of perverse coherent ideal sheaves
  of colength $= 1$ $\cong Y^+$: flop of $f$
  $\Rightarrow$ $D^b(Coh Y) \cong D^b(Coh Y^+)$ by Fourier--Mukai

---

**Def.**
1. $X$ = affine $P \in \text{Per}(Y/X)$ is a projective generator
   if it is a projective object in $\text{Per}(Y/X)$
   and $\text{Hom}_{\text{Per}(Y/X)}(P, E) = 0 \Rightarrow E = 0$
   
   e.g. $Y = \mathbb{C}^2 \to X = \mathbb{C}^2$ $P = \mathcal{O}_x \oplus \mathcal{O}_x(-1)$

2. In general, $P \in \text{Per}(Y/X)$ is a local projective generator
   if $X = \bigcup D_i$ open covering s.t. $\text{Per}_i$: projective generator

**Theorem** (Van der Berken)

\[
\begin{array}{ccc}
D^b(Coh Y) & \xrightarrow{\cong} & D^b(\text{mod}-(R\mathbb{P}^* \text{End}(P))) \\
\mathbb{P}^* \mathbb{R} \text{Hom}(P, \cdot) & \xleftarrow{\mathbb{P}^* \otimes^L P} & \mathbb{P}^* \mathbb{R} \text{Hom}(P, \cdot)
\end{array}
\]

\[\text{Per}(Y/X) \cong \text{mod } \mathbb{A}\]
We can construct moduli spaces of "stable" perverse coherent sheaves as moduli spaces of objects in $\text{mod} A$ via Simpson's method. (By a technical reason, we used a slight modification.)

**THEOREM (N+Yoshida)**

Consider the case $p: \hat{X} \to X$ blowup of a surface,
\& $\cdot$ support of $\text{R}^p_* \text{RHom}(P,E) : 2$-div.
\& coprime condition holds on $X$

(Stability $\iff$ semistability)

Then $E \in \text{Per}(\hat{X}/X)$ is stable ($\iff$ semistable)

\[ \iff \begin{array}{l}
\cdot E \in \text{Coh} \hat{X} \\
\cdot p^* E \text{ is torsion free and stable}
\end{array} \]

(In the framed) \[ \iff \begin{array}{l}
\text{O-stable} \\
\text{i.e.} \begin{array}{l}
\cdot \text{torsion free outside } C \\
\cdot \text{Hom}(E, \mathcal{O}_C(-1)) = 0, \text{Hom}(\mathcal{O}_C, E) = 0
\end{array}
\end{array} \]

:: moduli space of stable perverse coherent sheaves on $X$

$p^* \to$ moduli space of stable sheaves on $X$
$M$-stable $\iff$ $E(mC)$ is $O$-stable

Remark. $\text{Per}(X/X)$ is not preserved by $\otimes O(C)$.

We analyse what happens

$E$: $m$-stable but not $(m\pm 1)$-stable

$\Rightarrow$ wall-crossing

Technically we use a quiver description of the moduli space to apply the theory of the master space (just scheme does not work.)
Remark: We have a similar wall-crossing for \( p: Y \to X \) resolved conifold.

\[
\mathbb{Z}_{DT, Y} = \left( \prod_{m=1}^{\infty} (1 - (1-g)^m)^{-m} \right)^2 \times \left( \prod_{m=1}^{\infty} (1 - (1-g)^m)^{m} \right)
\]

\[
\mathbb{Z}_{NC,DT} = \left( \prod_{m=1}^{\infty} (1 - (1-g)^m)^{-m} \right) \times \left( \prod_{m=1}^{\infty} (1 - (1-g)^m) \right)^{\infty} \prod_{m=1}^{\infty} (1 - (1-g)^m)^{m} \]

\[
\mathbb{Z}_{PT, Y} = \prod_{m=1}^{\infty} (1 - (1-g)^m)^{m}
\]

\[
\mathbb{Z}_{PT, Y^+} = \prod_{m=1}^{\infty} (1 - (1-g)^m)^{-m}
\]

\[
\mathbb{Z}_{DT, Y^+} = \left( \prod_{m=1}^{\infty} (1 - (1-g)^m)^{-m} \right)^2 \times \left( \prod_{m=1}^{\infty} (1 - (1-g)^m)^{m} \right)
\]

\( \mathbb{Z}_{t_{inv}} \equiv 1 \)

Only \( \mathcal{O}_Y \) is stable.

\( \text{Per}(Y/X) = \text{Sendai's noncomm. DT} \)

\( \mathbb{Z}_{NC,DT} = \left( \prod_{m=1}^{\infty} (1 - (1-g)^m)^{-m} \right) \times \left( \prod_{m=1}^{\infty} (1 - (1-g)^m) \right)^{\infty} \prod_{m=1}^{\infty} (1 - (1-g)^m)^{m} \)

\( \mathbb{Z}_{PT, Y} = \prod_{m=1}^{\infty} (1 - (1-g)^m)^{m} \)

\( \mathbb{Z}_{PT, Y^+} = \prod_{m=1}^{\infty} (1 - (1-g)^m)^{-m} \)

\( \mathbb{Z}_{DT, Y^+} = \left( \prod_{m=1}^{\infty} (1 - (1-g)^m)^{-m} \right)^2 \times \left( \prod_{m=1}^{\infty} (1 - (1-g)^m)^{m} \right) \)

Example of Wall-crossing formula \& DT inv.

Since \( \mathcal{O}_{\text{c}}(-m) \) has no self-extensions, a direct calculation is very simple.

It is even simpler than the DT/stable pair wall explained by Thomas.