Quiver

$Q = (Q_0, Q_1)$

Q_0 : set of vertex, Q_1 : set of arrows

$h \in Q_1$ $\quad \overset{o(h)}{\circ} \longrightarrow \overset{i(h)}{\circ}$

$o(h)$: outgoing vertex
$i(h)$: incoming vertex

k : field (fixed)

A representation of Q is
- For each $i \in Q_0$, we have a f.d. k-vectr sp V_i
- For each $h \in Q_1$, we have a linear map

$$B_h : V_{o(h)} \longrightarrow V_{i(h)}$$

Ex (trivial) $V_i = 0 \ \forall i$, $B_h = 0 \ \forall h$
(direct sum) $V_i \oplus V_i'$, $\begin{bmatrix} B_h & 0 \\ 0 & B_h' \end{bmatrix}$

(V, B) : rep. of Q
(V', B')

$\rightsquigarrow \operatorname{Hom}_{\mathrm{rep}\,Q}((V,B),(V',B')) = \{ \xi = (\xi_i) \in \bigoplus \operatorname{Hom}(V_i, V_i')$

$$\begin{tikzcd} V_{o(h)} \arrow[r, "B_h"] \arrow[d, "\xi_{o(h)}"'] \arrow[dr, phantom, "\circlearrowright"] & V_{i(h)} \arrow[d, "\xi_{i(h)}"] \\ V'_{o(h)} \arrow[r, "B'_h"'] & V'_{i(h)} \end{tikzcd}$$

This is a k-vectr space

$\rightsquigarrow$ $\operatorname{Ker} \xi$, $\operatorname{Im} \xi$: rep. of Q in natural way

ξ : injective $\Rightarrow$ (V, B) : subrep. of (V', B')
surjective $\Rightarrow$ (V', B') : quotient rep. of (V, B)

Rep Q is an abelian category.

Def. Representations (V, B), (V', B') are isomorphic
$\Longleftrightarrow$ $\exists \xi \in \mathrm{Hom}_{\mathrm{rep}Q}((V,B),(V',B'))$

$\eta \in \mathrm{Hom}_{\mathrm{Rep}Q}((V',B'),(V,B))$ $\quad \xi\eta = \mathrm{id}$

Problem. Classify representations up to isomorphisms

① (Trivial) Case A_1 : $\circ$ with no arrow
$\Longrightarrow B = 0$ only $\dim V \in \mathbb{Z}_{\geq 0}$ is invariant

② Case A_2 : $\circ \to \circ$ $\quad V_1 \xrightarrow{B} V_2$

$V_1 \overset{\xi_1}{\cong} V_1'$ $\quad V_1' \xrightarrow{B'} V_2'$
$V_2 \underset{\xi_2}{\cong} V_2'$ s.th. $\xi_2 B \xi_1^{-1} = B'$

So it is equivalent to the problem:
classify matrices B up to $B \sim PBQ^{-1}$
$\rightarrow$ invariants are rank (and $\dim V_1$, $\dim V_2$)

③ Jordan quiver $\circlearrowleft\!\circ$ $\quad B$; square matrix
up to $B \sim PBP^{-1}$
∴ Jordan normal form $\leftarrow$ continuous parameter!
if $k = \overline{k}$

Def. A representation is indecomposable
$\Longleftrightarrow$ $(V, B) \not\cong (V', B') \oplus (V'', B'')$ and $\neq 0$

① ; Indecomposable $\Longleftrightarrow$ $\dim V = 1$

② ; Indecomposable $\Longleftrightarrow$ 3 types $\quad \mathbb{C} \longrightarrow 0$
$0 \longrightarrow \mathbb{C}$
or $\mathbb{C} \xrightarrow{\cong} \mathbb{C}$

③ : $\lambda \in k$ $\qquad B = \underbrace{\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}}_{n} \Big\} n$

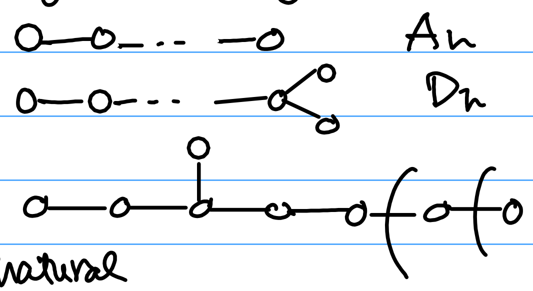
indecomposables $\longleftrightarrow \{ (\lambda, n) \in k \times \mathbb{Z}_{>0} \}$

<u>Report Problem 1</u>
Classify indecomposable representations of Kronecker quiver $\circ \rightrightarrows \circ$
$(k = \bar{k})$

<u>Finite type</u> Q is of finite type
$\underset{\text{def.}}{\iff}$ There is only finitely many indecomposable representations

<u>ex.</u> A_1, A_2 : finite type $\longrightarrow$ finite type
Jordan (Kronecker) quiver $\longrightarrow$ nonfinite type

<u>Th</u> [Gabriel, Bernstein-Gelfand-Ponomarev]
Q is finite type $\iff$ If we forget orientations of arrows and replace $\circ \to \circ$ by $\circ - \circ$, we have ADE Dynkin diagram


A_n
D_n
$E_{8,7,6}$

Moreover indecomposable rep. $\underset{\text{bijection}}{\overset{\text{natural}}{\longleftrightarrow}}$ positive roots of the corresponding Lie alg.

e.g. $A_1 = \mathfrak{sl}_2(\mathbb{C}) = \{\text{tracefree } 2\times 2 \text{ matrices}\}$

$A_2 = \mathfrak{sl}_3(\mathbb{C}) = \{\quad " \quad 3\times 3 \quad " \quad\}$

$$\mathfrak{sl}_3(\mathbb{C}) = \left\{ \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \middle| \ x_1+x_2+x_3=0 \right\} \leftarrow \text{Cartan subalgebra } [A,B]=0$$

$$\oplus \begin{bmatrix} 0 & * & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ & 0 & * \\ & & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & * \\ & 0 & 0 \\ & & 0 \end{bmatrix} \oplus \text{ similar for lower triangular}$$

3 positive root subspaces

$$\left[\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix} \right] = (x_1 - x_2) \begin{bmatrix} 0 & * & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

$$\left[\quad " \quad, \begin{bmatrix} 0 & & \\ & 0 & * \\ & & 0 \end{bmatrix} \right] = (x_2 - x_3) \begin{bmatrix} & & \\ & & * \\ & & \end{bmatrix}$$

$$\left[\quad " \quad, \begin{bmatrix} 0 & 0 & * \\ & 0 & 0 \\ & & 0 \end{bmatrix} \right] = (x_1 - x_3) \begin{bmatrix} & & * \\ & & \\ & & \end{bmatrix}$$

$Q \rightsquigarrow$ Tits form $\quad x \in \mathbb{Z}^{Q_0}$

$$(x, x) \overset{\text{def.}}{=} \sum_{i \in Q_0} x_i^2 - \sum_{h \in Q_1} x_{o(h)} x_{i(h)}$$

<u>Ex.</u> $\circ \to \circ \quad A_2 \quad x_1^2 + x_2^2 - x_1 x_2 \quad$ +ve definite

<u>Report Problem 2</u>

Q is of finite type $\Longleftrightarrow$ $(\ ,\ )$ is positive definite

Geometric Proof of $\Longrightarrow$ of Gabriel ...'s thm

Fix V

$$\bigoplus \mathrm{End}(V_i) \underset{\text{open}}{\subset} G(V) = \prod_{i \in Q_0} GL(V_i)$$

$$\mathbb{E}(V) = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \curvearrowleft G(V)$$

$G(V)$-orbits $\longleftrightarrow$ isomorphism classes of quiver rep's (of given dim's)

$$\dim(\text{orbits}) = \dim G(V) - \dim(\underbrace{\text{stabilizer}}_{\{\xi \in G(V) \mid \xi B \xi^{-1} = B\}})$$

$$= \underbrace{\sum_{i \in Q_0} \dim V_i^2}_{1^{st} \text{ term of } (\ ,\ )} - \dim(\text{stabilizer})$$

On the other hand,

$$\dim \mathbb{E}(V) = \underbrace{\sum_{h \in Q_1} \dim V_{o(h)} \dim V_{i(h)}}_{2^{nd} \text{ term of } (\ ,\ )}$$

Only finitely many indecomposables $\Longrightarrow$ Only finitely many orbits

$\therefore$ $\exists$ maximal dim. orbit $=$ $\dim \mathbb{E}(V)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \ \|$

$\therefore$ $\dim G(V) - \dim(\text{stabilizer})$, $\quad \text{stabilizer} \supset \mathbb{R}^\times$

$\therefore$ $\dim G(V) - \dim \mathbb{E}(V) = \dim(\text{stabilizer}) \geq 1$

$\therefore$ $(\ ,\ )$ is positive definite. //