https://www.dropbox.com/s/o2ck498f1u5buqw/2020-09-28.pdf?dl=0

Aim quantized universal enveloping algebra

$U_q(\mathfrak{g}) \supset U_q^-(\mathfrak{g})$: lower triangular part

canonical base (Lusztig)

= global crystal base (Kashiwara)

$$\begin{tikzcd} \text{PBW base} \arrow[r, bend right, "\text{elementary approach}"'] & \text{canonical base} \end{tikzcd}$$

relation to representation theory of quivers

§ q-integers

q: variable

q-integer $[n] \overset{\text{def}}{=} \dfrac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$
$\in \mathbb{Z}[q, q^{-1}] =: A$

$\mathbb{F} := \mathbb{Q}(q)$

ring hom $\begin{matrix} A & \longrightarrow & \mathbb{Z} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ q & \longmapsto & 1 \end{matrix}$ $\quad \underset{q\text{-analog}}{[n]} \mapsto \underset{\text{classical}}{n}$

q-factorial : $[n]! = [n] \cdots [1]$

We do not consider a specialization at root of unity in these lectures.

Suppose $\zeta = \exp(2\pi\sqrt{-1}/n)$ primitive n^{th} root of unity

$\Longrightarrow \begin{matrix} A & \longrightarrow & \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ q & \longmapsto & \zeta \end{matrix}$ sends $[n] \mapsto 0$

more generally $[n+k] = [k]$ in the image

This is similar to $\mathbb{F}_p$ if $p = n$

representation theory of QUE at $q = \zeta$ is similar to
$n=p$
repn theory of reductive groups in char $= p$

§ $U_q(\mathfrak{sl}_2)$

Recall $\underset{\in}{\mathfrak{sl}_2} = \{ X : 2\times 2 \text{ cpx matrix}, \operatorname{tr} X = 0 \}$

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{✩} \quad \begin{array}{l} [h, e] = 2e \\ [h, f] = -2f \end{array} \quad , \; [e, f] = h$$

$\mathfrak{sl}_2 \cong$ Lie algebra generated by e, f, h
with the defining relation ✩

universal enveloping algebra $U(\mathfrak{sl}_2)$
$=$ universal object among $\mathfrak{g} \xrightarrow{\text{Lie alg. hom}} A$ associative algebra

i.e. any other A'

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr, "\forall"'] & A \arrow[d, "\exists \text{ alg. hom}"] \\ & A' \end{tikzcd}$$

$\cong$ ass. alg. generated by e, f, h
with the defining relation ✩

Def. $U_q(\mathfrak{sl}_2)$ is an associative algebra $/ \underset{\substack{\| \\ \mathbb{F}}}{\mathbb{Q}(q)}$

generators $\quad e, f, t, t^{-1}$

relations $\quad tt^{-1} = t^{-1}t = 1$

$$te t^{-1} = q^2 e$$

$$tf t^{-1} = q^{-2} f$$

$$[e, f] = \frac{t - t^{-1}}{q - q^{-1}}$$

Roughly $t = q^h$. Taking limit $q \to 1$

$$\frac{t - t^{-1}}{q - q^{-1}} = \frac{q^h - q^{-h}}{q - q^{-1}} \longrightarrow h$$

$U^+ = \langle e \rangle \leftarrow$ has linear base $1, e, e^2, \cdots$
$U^- = \langle f \rangle$ $\quad 1, f, f^2,$
$U^0 = \langle t^{\pm} \rangle = \mathbb{Q}(q)[t, t^{-1}]$

easy to check
<u>Th</u> (cf. Jantzen Th1.5)
$e^k t^n f^l$ $\quad k, l \geq 0 \quad n$: arbitrary
forms a $\mathbb{Q}(q)$ linear base of $U_q(sl_2)$

<u>Cor</u>. $U_q(sl_2) \underset{\text{multiplication}}{\leftarrow\!\!\!\cong} U^+ \otimes U^0 \otimes U^-$ as $\mathbb{Q}(q)$-vector space

This, at least, says $U^{\pm}$ has the same size with $U(\langle e \rangle)$, $U(\langle f \rangle)$

U^0 : more like the space of functions on torus in SL_2
$$\shortparallel$$
$$\left\{ \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \middle| \, x \in \mathbb{C}^\times \right\}$$

$t^{\pm 1} \leftrightarrow$ function $\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \mapsto x^{\pm 1}$

§ representations of $U_q(\mathfrak{sl}_2)$
modules

○ look for 1-dim. representations
$$V = \mathbb{Q}(q)u = \mathbb{F}u$$

All operators act by scalar ⇒ commute each other

$$\therefore \quad eu = tet^{-1}u = q^2 eu \qquad \therefore eu = 0$$
Similarly $fu = 0$

$$\therefore 0 = [e, f]u = \frac{t - t^{-1}}{q - q^{-1}} u \qquad \therefore t \text{ acts by } \pm 1$$

Conversely
$$e \mapsto 0$$
$$f \mapsto 0$$
$$t \mapsto \pm 1$$
is a 1-dim'l rep. on $V = \mathbb{F}u$

○ Mimicking finite dimensional representations of $\mathfrak{sl}_2$, we consider

$$V(\pm q^n) = \mathbb{F}v_0 \oplus \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_n$$

$$\begin{tikzcd} v_0 \arrow[loop below, "t"] \arrow[r, shift left, "f"] & v_1 \arrow[l, shift left, "e"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & v_n \arrow[l, shift left] \end{tikzcd}$$

[谷崎 1.4.1]
$$\left(\begin{array}{l} tv_i = \pm q^{n-2i} v_i \\ ev_i = \pm [n-i+1] v_{i-1} \\ fv_i = [i+1] v_{i+1} \end{array} \right. \qquad \left(v_i = \frac{f^i}{[i]!} v_0 \right)$$

<u>Lemma</u> $[i+1][n-i] - [n-i+1][i] = [n-2i]$

This defines an $(n+1)$-dimensional representation

Def. A finite dim'l rep. of $U_q(\mathfrak{sl}_2)$ is of type I $\underset{\text{def.}}{\Longleftrightarrow}$ eigenvalues of t $\in q^{\mathbb{Z}}$

More generally for $\lambda \in \mathbb{Q}(q)^{\times}$, we define $v_0, v_1 \cdots$

$$\begin{cases} t^{\pm} v_i = \lambda^{\pm} q^{\mp 2i} v_i \\ e v_i = \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} v_{i-1} \\ f v_i = [i+1] v_{i+1} \end{cases}$$

$$(\lambda q^{-i} - \lambda^{-1} q^{i})(q^{i+1} - q^{-i-1}) - (\lambda q^{1-i} - \lambda^{-1} q^{i-1})(q^{i} - q^{-i})$$

$$= \cancel{\lambda q} - \lambda^{-1} q^{2i+1} - \lambda q^{-2i-1} + \cancel{\lambda^{-1} q^{-1}}$$

$$- \cancel{\lambda q} + \lambda^{-1} q^{2i-1} + \lambda q^{1-2i} - \cancel{\lambda^{-1} q^{-1}}$$

$$= \lambda^{-1} q^{2i}(-q + q^{-1}) + \lambda q^{-2i}(q - q^{-1})$$

This is called a Verma module $M(\lambda)$.
Write v_λ instead of v_0 hereafter.

Prop. M: $U_q(\mathfrak{sl}_2)$-module s.t. $\exists m \in M$ $em = 0$, $tm = \lambda m$

$$\Longrightarrow \exists 1 \quad \varphi : \begin{array}{ccc} M(\lambda) & \longrightarrow & M \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_\lambda & \longmapsto & m \end{array} \quad U_q(\mathfrak{sl}_2)\text{-hom}$$

The previous $(n+1)$-dim'l rep. (of type I) is a quotient of $M(q^n) \twoheadrightarrow V(q^n)$

$$\lambda = q^n \Rightarrow \frac{\lambda q^{-n} - \lambda^{-1} q^n}{q - q^{-1}} = 0$$

$\therefore$ $v_{n+1}, v_{n+2}, \cdots$ is a submodule of $M(q^n)$

$0 \xleftarrow{e} v_{n+1}$

$= \operatorname{Ker}$ of $M(q^n) \twoheadrightarrow V$.

<u>Prop</u>. $V(q^n)$ is irreducible [谷崎 1.45]

$\because$ $S \subset V$ submodule

$tS \subset S$ eigenvalues are distinct

$\Longrightarrow$ S is spanned by v_i

Suppose $v_i \in S$.

$e, f(S) \subset S \Rightarrow v_{i\pm1}$ is also in S

$\therefore S = 0$ or $S = V$ //

谷1.62

<u>Prop</u>. Any fin. dim irr. rep. $V \cong V(\pm q^n)$ for some n

$\because$ Consider $V \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ ← alg. closure

$\ni v$

and take v : eigenvector for e eigenvalue λ

$$e \underline{t^k v} = q^{-2k} t^k e v = \lambda q^{-2k} \underline{t^k v}$$

$\therefore \lambda q^{-2k}$ is also eigenvalue

all distinct unless $\lambda = 0$

finite dim. $\Rightarrow$ $\lambda = 0$

. . .

Consider $\{ v \in V \otimes_{\mathbb{F}} \overline{\mathbb{F}} \mid ev = 0 \}$

$\circlearrowleft$ inv. under t

$\therefore$ $\exists$ eigenvector v with eigenvalue λ

$$\therefore \quad \begin{array}{ccc} M(\lambda) & \longrightarrow & V \otimes_{\mathbb{F}} \overline{\mathbb{F}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_\lambda & \longmapsto & v \end{array}$$

But if $\dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} \neq 0 \quad \forall i$

eigenvalue λq^{-2i}

$\Rightarrow$ $v, fv, f^2 v, \ldots$ all nonzero and linearly indep (eigenvalues are distinct)

$\therefore \quad \lambda \in \pm q^{\mathbb{Z}}$

Now we can take v in V.

$$\text{and} \quad \begin{array}{ccc} V(\pm q^n) & \longrightarrow & V \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_0 & \longmapsto & v \end{array}$$

- injective by above conside
- surjective by irreducibility

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Hopf algebra

A : alg / k k : field

$m : A \otimes_k A \longrightarrow A$ mult.

$u : k \longrightarrow A$ unit

associativity :

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\otimes 1"'] \arrow[d, "1\otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

unit

$$\begin{tikzcd}
k\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\cong", "\text{nat. iso}"'] & A\otimes A \arrow[r] \arrow[d, "m"] & A\otimes k \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

coalgebra : reverse the directions of arrows

$\Delta : A \to A\otimes A$ comultiplication

$\varepsilon : A \to k$ counit

Def. A is bialg. $\Longleftrightarrow$ Δ, ε are alg. hom

where $A\otimes A$ $(a\otimes b)\cdot(c\otimes d) = ac\otimes bd$

antipode $S : A \to A$

s.t.

$$\begin{tikzcd}
A\otimes A \arrow[d, "S\otimes 1"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "u\circ\varepsilon"] & A\otimes A \arrow[d, "1\otimes S"] \\
A\otimes A \arrow[r, "m"] & A & A\otimes A \arrow[l, "m"']
\end{tikzcd}$$

$$\Delta x = \sum_i x_i' \otimes x_i'' \Rightarrow \sum S(x_i')\, x_i'' = u\circ\varepsilon(x)$$

e.g. $A = U(\mathfrak{g})$ $\mathfrak{g}$: Lie algebra

- $\Delta(X) = X\otimes 1 + 1\otimes X$ $X \in \mathfrak{g}$

$$\left(\begin{aligned} \Delta(XY) &= (X\otimes 1 + 1\otimes X)(Y\otimes 1 + 1\otimes Y) \\ &= XY\otimes 1 + Y\otimes X + X\otimes Y + 1\otimes XY \\ \Delta(YX) &= X \leftrightarrow Y \end{aligned} \right.$$

$\Rightarrow$

- $\varepsilon(X) = 0$
- $S(X) = -X$ & S: anti alg. hom

$$m(S\otimes 1)\Delta(X) = m(\underbrace{S(X)}_{\overset{\shortparallel}{-X}}\otimes 1 + 1\otimes X) = 0$$

Fact S is anti-auto automatically

A: Hopf alg. $\Longrightarrow$ • $V\otimes W$ is an A-module
V, W: A-module via Δ

- $V^* = \mathrm{Hom}(V, k)$ is an A-module via S

NB. $(V^*)^* \not\cong V$ in general unless $S^2 = id$

- k is an A-module via ε

Claim natural pairing
$V^*\otimes V \longrightarrow k$ is an A-module hom.

$\because$ $\Delta x = \sum_i x_i'\otimes x_i''$

$$x(f\otimes v) = \sum f\circ S(x_i')\otimes x_i'' v \xmapsto{\text{pairing}}$$

$$\mapsto f(\underbrace{\sum S(x_i')x_i''}_{\overset{\shortparallel}{u\varepsilon(x)}} v) = \varepsilon(x) f(v)$$

//

V, W : A-module

Hom(V, W) is an $A \otimes A^{op}$-module

by $(x' \otimes x'')f = x' f(x'' \cdot)$

composing $(\mathrm{Id} \otimes S) \circ \Delta$, we get an A-module structure on Hom(V, W).

By definition $\mathrm{Hom}(V, W) \cong W \otimes V^*$, as

$$(\mathrm{Id} \otimes S)\Delta(x) = \sum x_i' \otimes S(x_i'')$$

Claim $k \longrightarrow \mathrm{Hom}(V, V)$ is an A-module hom

$$\begin{tikzcd} k \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \arrow[r] & \mathrm{Hom}(V,V) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ 1 \arrow[r, mapsto] & \mathrm{id} \end{tikzcd}$$

∵ $x \cdot \mathrm{id} = \sum x_i' \, \mathrm{Id}(S(x_i'') \cdot)$

$$= \sum_i x_i' S(x_i'') \cdot = u \circ \varepsilon(x) \cdot \quad //$$

$U_q(sl_2)$ is a Hopf algebra by the following :

$$\Delta e = e \otimes t^{-1} + 1 \otimes e$$
$$\Delta f = f \otimes 1 + t \otimes f \qquad \Delta(t^{\pm}) = t^{\pm} \otimes t^{\pm}$$
$$\varepsilon(t) = 1, \quad \varepsilon(e), \varepsilon(f) = 0$$
$$S(t) = t^{-1}, \quad S(e) = -et, \quad S(f) = -t^{-1}f$$

<u>Def.</u> A is commutative alg. $\Longleftrightarrow$

$$\begin{tikzcd} A\otimes A \arrow[r, "swap"] \arrow[dr, "m"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowright"] \\ & A \end{tikzcd}$$

A is cocommutative $\Longleftrightarrow$

$$\begin{tikzcd} A\otimes A & A\otimes A \arrow[l, "swap"'] \arrow[dl, phantom, "\circlearrowright"] \\ & A \arrow[u, "\Delta"'] \arrow[ul, "\Delta"] \end{tikzcd}$$

$U(\mathfrak{g})$ is cocommutative (noncommutative)

But $U_q(sl_2)$ is neither comm. nor cocommutative.

○ Complete reducibility (谷. Th 1.65)

Def. $C = fe + \frac{qt + q^{-1}t^{-1}}{(q-q^{-1})^2} \in U_q(sl_2)$

(Casimir element)

Lemma (1) $C \in$ center $U_q(sl_2)$

(2) C acts on $M(\lambda)$ by scalar mults by $\frac{\lambda q + \lambda^{-1}q^{-1}}{(q-q^{-1})^2}$

∵ (1) direct calculation

(2) $Cv_\lambda = (\text{above})\, v_\lambda$

$C f^n v_\lambda = f^n C v_\lambda = (\text{above})\, f^n v_\lambda$ //

Th. finite dim'l representations of $U_q(sl_2)$ are completely reducible, i.e. direct sum of irreducible representations

∵ Using Casimir elements, we can argue as for sl_2-representations //