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Correction

A: Hopf algebra $\qquad S$: antipode

Recall $\circ$ $V^* \otimes V \longrightarrow k$ $\qquad$ A-alg. hom

another axiom $\Longleftrightarrow$ $k \longrightarrow \mathrm{Hom}(V,V)$ A-alg. hom

$\mathrm{Hom}(V,W) \cong W \otimes V^*$

A-alg. hom

$U_q(\mathfrak{sl}_2) = \langle e, f, t^{\pm} \rangle / \text{relation}$

Hopf alg $\quad \Delta$: coproduct $\quad U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$

- finite dim rep. of $U_q(\mathfrak{sl}_2)$ (type I) $\longleftrightarrow$ f.d. rep. of $\mathfrak{sl}_2$
- complete reducibility
- $\Delta \neq \mathrm{swap} \circ \Delta \qquad \mathrm{swap}(a' \otimes a'') = a'' \otimes a'$

In particular $\quad V \otimes W \not\cong W \otimes V$

More precisely $\quad v \otimes w \longmapsto w \otimes v$
is <u>not</u> an A-hom.
$U_q(\mathfrak{sl}_2)$

Corresponding $\mathfrak{sl}_2$-rep. $V \otimes W \cong W \otimes V$

Above $\Longrightarrow$ $V \otimes W \cong W \otimes V$ as abstract $U_q(\mathfrak{sl}_2)$-modules

This isom. is <u>not</u> swap!

The <u>correct</u> $U_q(\mathfrak{sl}_2)$-module isom. is given by <u>R-matrix</u>. (explicit)

universal R-matrix $\mathcal{R} \in U_q(\mathfrak{sl}_2) \hat{\otimes} U_q(\mathfrak{sl}_2)$ ← completion

invertible (indep. of repr.)

s.t. $V \otimes W \xrightarrow[\cong]{\mathcal{R} \circ \text{swap}} W \otimes V$

$\mathcal{R}$ is well-defined (∞-sum $\rightarrow$ finite sum)

<u>Exercise</u> Give details of the following computation.

$n \in \mathbb{Z}_{\geq 0}$

Define $\Theta_n := a_n e^n \otimes f^n$ where $a_n = q^{\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]!}$

<u>Rem</u> $q \in \sqrt[\circ]{1} \Rightarrow [n]! = 0$

$\Rightarrow a_n$ is <u>not</u> well-defined

<u>Claim</u> (1) $(f\otimes 1)\Theta_n + (t\otimes 1)\Theta_{n-1}$
$$= \Theta_n (f\otimes 1) + \Theta_{n-1}(t^{-1}\otimes f)$$
and similar formula for e

(2) $(t\otimes t)\Theta_n = \Theta_n (t\otimes t) \leftarrow \mathrm{wt}\,\Theta_n = 0$

$$\Theta = \sum_{n=0}^{\infty} \Theta_n \quad \in \quad U_q(\mathfrak{sl}_2)\hat{\otimes} U_q(\mathfrak{sl}_2)$$

<u>Note</u> On f.d. rep., e^n, f^n act by 0 for $n \gg 0$.

Furthermore consider $q^{\frac{1}{2}H\otimes H}$ $\left(= \sqrt{t\otimes t}\right)$:

$V\otimes W \ni v\otimes w$ $\qquad tv = q^n v$, $tw = q^m w$

$$q^{\frac{1}{2}H\otimes H}(v\otimes w) := q^{\frac{nm}{2}}\, v\otimes w$$

<u>Claim</u> $q^{\frac{1}{2}H\otimes H}(1\otimes e) = (t\otimes e)\, q^{\frac{1}{2}H\otimes H}$

$\because$ Apply to $V\otimes W$: $q^{\frac{n(m+2)}{2}}$ vs $q^n \times q^{\frac{nm}{2}}$

$=$ //

We define $\mathcal{R} \overset{\text{def}}{=} \sum_{n=0}^{\infty} \Theta_n\, q^{\frac{1}{2}H\otimes H}$

($q^{\frac{1}{2}}$ is <u>not</u> a problem when applied to repr.)

Th. $\Delta(u)\,\mathcal{R} = \mathcal{R}\circ \mathrm{swap}\circ \Delta(u) \quad \forall u \in U_q(\mathfrak{sl}_2)$

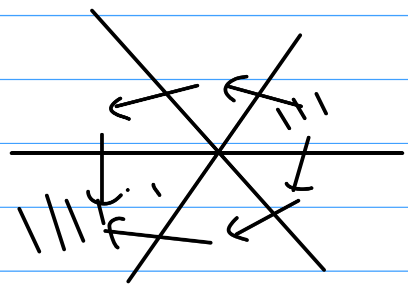
$\mathcal{R}$: universal R-matrix

Exercise (easy): Compute $\mathcal{R}$ for $\begin{pmatrix}\text{2dim} \\ \text{rep.}\end{pmatrix} \otimes \begin{pmatrix}\text{2dim} \\ \text{rep.}\end{pmatrix}$

(4×4 matrix)

Yang-Baxter equation:

$$\begin{tikzcd}
\underline{V_1\otimes V_2\otimes V_3} \arrow[r] \arrow[d] & V_2\otimes V_1\otimes V_3 \arrow[r] \arrow[d, phantom, "\circlearrowright"] & \bigcirc \arrow[d] \\
\bigcirc \arrow[r] & \bigcirc \arrow[r] & \underline{V_3\otimes V_2\otimes V_1}
\end{tikzcd}$$



cf. $s_1 s_2 s_1 = s_2 s_1 s_2$ in $W(\mathfrak{sl}_3)$

$\rightsquigarrow$ braid group vs R-matrix

$\rightsquigarrow$ knot invariants from $U_q(\mathfrak{sl}_2)$

eg. Jones polynomials
Reshetikhin–Turaev

$\rightsquigarrow$ 3dim manifold inv. (Witten)

§ $U_q(\mathfrak{g})$ $\quad \mathfrak{g}$: symmetrizable Kac-Moody Lie alg.

(In practice, we only consider finite dim complex, simple Lie algebras)

I : index set $\quad \# I < \infty$

$A = (a_{ij})_{i,j \in I}$: Cartan matrix

$$\overset{\text{def.}}{\iff} \begin{cases} a_{ii} = 2 \\ a_{ij} \leqq 0 \quad \text{if } i \neq j \\ a_{ij} = 0 \iff a_{ji} = 0 \end{cases} \qquad a_{ij} \in \mathbb{Z}$$

symmetrizable $\iff {}^{\exists} d_i \neq 0$ s.t. $d_i a_{ij} = d_j a_{ji}$

(symmetrizer $(d_i \in \mathbb{Z}_{>0})$

Dynkin diagram

$$\begin{array}{ll} \overset{i}{\circ}\text{———}\overset{j}{\circ} & \text{if } a_{ij} = a_{ji} = -1 \\ \circ \qquad \circ & \text{if } a_{ij} = a_{ji} = 0 \\ \circ \Longrightarrow \circ & \text{if } a_{ij} = -1, a_{ji} = -2 \\ \circ \Rrightarrow \circ & \qquad\qquad\qquad\qquad -3 \end{array}$$

$$\left(\begin{array}{ll} \circ \overset{|a_{ij}|}{\text{———}} \circ & \text{if } a_{ij} = a_{ji} \text{ \& } |a_{ij}| > 1 \\ \circ \overset{|-a_{ij}| \quad |-a_{ji}|}{\text{———}} \circ & \text{if } a_{ij} \neq a_{ji} \end{array} \right.$$

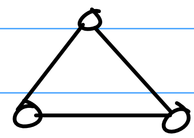
Def. A is indecomposable $\iff$ Dynkin diagram is connected

Assume A is indecomposable hereafter.

A is of finite type $\overset{\text{def.}}{\iff}$ symmetric matrix given by $d_i a_{ij} = d_j a_{ji}$ is <u>positive definite</u>

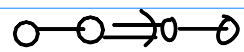
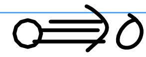
$\rightsquigarrow$ use inner product $(\ ,\ )$ later

affine type $\overset{\text{def.}}{\iff}$ positive semi-definite, $\operatorname{rank} A = \operatorname{size} A - 1$

(e.g.  $A_2^{(1)}$ $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Classification of cpx simple Lie alg.
$\leftrightarrow$ classif. of Cartan matrix of finite type

$A_n, B_n, C_n, D_n, E_{6,7,8}, F_4, G_2$

affine $A_n^{(1)}, \cdots$

We define Kac-Moody Lie algebras $\mathfrak{g}(A)$ associated with Cartan matrix A

A : finite type $\Longrightarrow$ $\mathfrak{g}(A)$: cpx simple Lie algebra

Serre relation (one of the defining relation in $\mathfrak{g}(A)$)

e.g. $\mathfrak{sl}_n(\mathbb{C})$
A_{n-1}

$$e_i = \begin{array}{c|c} & i+1 \\ i & \begin{bmatrix} \cdots\cdots & 1 & \\ & & \\ & & \end{bmatrix} \end{array}$$

$$[e_{i-1}, e_i] = \begin{bmatrix} & & 1 \\ & & \\ & & \end{bmatrix}$$

$$[e_{i-1}, [e_{i-1}, e_i]] = 0$$

$$\left(\left(\begin{array}{l} \mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g}) \\ \quad\quad \| \\ \quad\quad [x, \cdot] \end{array} \right.\right.$$

$$\mathrm{ad}(e_{i-1})^2 e_i = 0$$

Expand as $e_{i-1}^2 e_i - 2e_{i-1}e_ie_{i-1} + e_ie_{i-1}^2 = 0$

More generally

$$(\mathrm{ad}\, x)^n(y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k y x^{n-k}$$

One of defining relations of QUE : quantum Serre relation

deformation of $(\quad) = 0$

$\underline{\text{Def}}$. A Kac-Moody Lie alg $\mathfrak{g} = \mathfrak{g}(A)$ (/ $\mathbb{Q}$)

(Assume A is full rank) $\times$ affine

is

generators $\quad e_i, f_i, h_i \quad (i \in I)$

relations

Serre relation

$$\operatorname{ad}(e_i)^{1-a_{ij}}(e_j) = 0$$
$$\operatorname{ad}(f_i)^{1-a_{ij}}(f_j) = 0$$

+ easier rel.

$$[h_i, h_j] = 0$$
$$[h_i, e_j] = a_{ij} e_j$$
$$[h_i, f_j] = -a_{ij} f_j$$
$$[e_i, f_j] = \delta_{ij} h_i$$

$\mathfrak{h}$: Cartan subalgebra $= \bigoplus_{i \in I} \mathbb{Q} h_i \subset \mathfrak{g}$

$[h_i, h_j] = 0$ commutative subalgebra

$\underline{\text{Rem}}$. If A is $\underline{\text{not}}$ full rank, we first define $\mathfrak{h}$

s.t. $\dim \mathfrak{h} = 2|I| - \operatorname{rank} A$ (e.g. affine type $|I| + 1$)

Choose

$h_i \in \mathfrak{h}$ s.t. $\underline{\text{linearly independent}}$

$\langle e_i, f_i, h_i \rangle \subset \mathfrak{g}$

$\cong$ $\mathfrak{sl}_2$

easier rel

By $\mathrm{ad}\, x$, $\mathfrak{g}$ is regarded as an $\mathfrak{sl}_2$-module.

Take f_j $(j \neq i)$ and consider

$\mathfrak{sl}_2$-submodules V generated by f_j

easier rel. : $[e_i, f_j] = 0$, $[h_i, f_j] = -a_{ij} f_j$

$$V = \sum_{k \geq 0} \mathbb{Q} f_i^k \cdot f_j \;\; \exists \twoheadleftarrow M(\underbrace{-a_{ij}}_{\in \mathbb{Z}_{\geq 0}})$$

Verma module

V is finite dim'l $\mathfrak{sl}_2$-module

$\Longleftrightarrow (\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j) = 0$

$$\underset{\substack{\circlearrowleft \\ h_i = -a_{ij} \times}}{\overset{f_j}{\bullet}} \xrightarrow{f_i} \bullet \longrightarrow \bullet \; \cdots \; \overset{f_i^{-a_{ij}}(f_j)}{\bullet} \longrightarrow = 0$$

$\therefore$ Serre relation $\Longleftrightarrow$ V : finite dim'l

Prop [命 2.25] (Serre rel. $\Rightarrow$) $\mathfrak{sl}_2 \curvearrowright \mathfrak{g}$ direct sum of f.d. repr. of $\mathfrak{sl}_2$

☆ PBW base of $U(\mathfrak{g})$ = universal enveloping algebra of $\mathfrak{g}$

For later purposes, we first recall:
roots. Weyl group, reduced expression.

$\mathfrak{h}$: Cartan subalg. $= \bigoplus \mathbb{Q} h_i$

$\mathfrak{n} = \langle e_i \rangle$ Then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$
$\mathfrak{n}^- = \langle f_i \rangle$

root : decompose $\mathfrak{g}$ into simult. eigenspaces w.r.t $\mathfrak{h}$.

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [h, X] = \langle \alpha, h \rangle X \ \ \forall h \in \mathfrak{h} \}$$

$$\alpha \in \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{Q})$$

root subspace

$$\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha = \underset{\substack{\shortparallel \\ (\mathfrak{g}_0)}}{\mathfrak{h}} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$$

$$\alpha : \text{root} \overset{\text{df.}}{\iff} \alpha \neq 0, \ \mathfrak{g}_\alpha \neq 0$$

Define $\alpha_i \in \mathfrak{h}^*$ (simple root) by $\alpha_i(h_j) = a_{ji}$

e.g. $\mathfrak{g}_{\alpha_i} = \mathbb{Q} e_i$

$\mathfrak{g}_{-\alpha_i} = \mathbb{Q} f_i$

<u>Fact</u> $\mathfrak{g}_\alpha \subset \mathfrak{n}$ or $\mathfrak{n}^-$
$\Rightarrow$ root positive $(\mathfrak{n})$, negative $(\mathfrak{n}^-)$

positive $\Leftrightarrow$ $\alpha = \sum m_i \alpha_i \quad m_i \in \mathbb{Z}_{\geq 0}$

negative $\Leftrightarrow$ $-\alpha$: positive

Δ = set of roots

Δ^+ = 〃 positive 〃

Δ^- = 〃 negative 〃

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$$

Recall $\langle f_j, [f_i, f_j], \ldots, (\mathrm{ad} f_i)^{-a_{ij}}(f_j) \rangle \xrightarrow{f_i} = 0$

$\uparrow$ $\mathfrak{sl}_2$- submodule $\uparrow$

$\uparrow$

$\alpha_j, \ \alpha_j + \alpha_i, \ \ldots, \ -a_{ij}\alpha_i + \alpha_j$

$(a_{ij} = -1 \Rightarrow \alpha_i + \alpha_j)$

$(\phantom{a_{ij} =} -2 \qquad 2\alpha_i + \alpha_j)$

root

$\mathfrak{sl}_2$- rot has symmetry

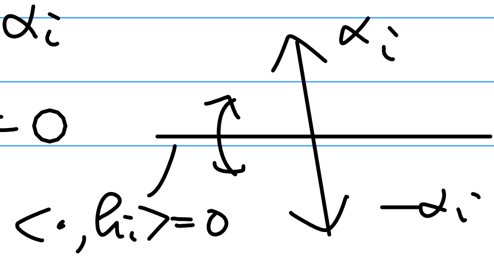
-1

Define simple reflection: $s_i \in GL(\mathfrak{h}^*)$ by

$$s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$$

i.e. $s_i(\lambda) = \lambda$ if $\langle \lambda, h_i \rangle = 0$

$s_i(\alpha_i) = -\alpha_i$



In part. $s_i^2 = 1$

<u>Def.</u> Weyl group $W = \langle s_i \rangle \subset GL(\mathfrak{h}^*)$

In general, W cannot act on $\mathfrak{g}$

$\underline{\mathfrak{sl}_2} \rightsquigarrow SL_2 \ni \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ exchanges e and f

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overset{e}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} & \end{bmatrix}^{-1} = \underset{-f}{\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}}$

$\parallel$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \underbrace{\exp(\mathrm{ad}\, e_i) \exp(-\mathrm{ad}\, f_i) \times \exp(\mathrm{ad}\, e_i)}_{\overset{\parallel}{\tau_i}}$

τ_i is a well-defined operator acting on $\mathfrak{g}$.

By definition $\tau_i : \mathfrak{g}_\alpha \xrightarrow{\cong} \mathfrak{g}_{s_i(\alpha)}$

$\tau_i^2 = -1 \ (\neq 1)$

$\rightsquigarrow$ a finite covering of W acts on $\mathfrak{g}$.

<u>Fact</u> ① W : finite group $\iff$ A : finite type

$(\Leftarrow)$ $W \subset GL_{\mathbb{Z}}(\mathfrak{h}^*) \cap O(\mathfrak{h}^*)$

compact as pos. definite.

② $\forall \alpha \in \Delta$ is obtained from a simple root α_i
$\Vert$ by applying an element w
$w(\alpha_i)$ of W.

(For non-finite type KM Lie alg
$\Rightarrow$ this is ~~not true~~,
imaginary root

Cor. finite type $\quad \mathfrak{g}_\alpha \cong \mathfrak{g}_{\alpha_i} = \mathbb{Q} e_i$

$$\underset{\substack{\Vert \\ w\alpha_i}}{\mathfrak{g}_\alpha}$$

$\Rightarrow \dim \mathfrak{g}_\alpha = 1$ (root multiplicity >1 for imaginary root)