Recall $\mathfrak{g} = \langle e_i, f_i, h_i \rangle$: Kac-Moody

$\alpha_i \in \mathfrak{h}^*$ simple root

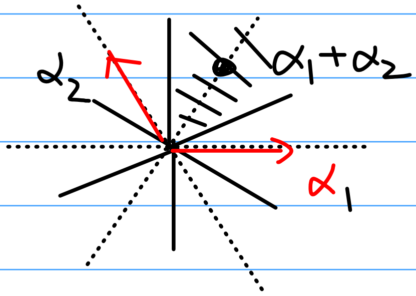
$$\Delta = \{ \alpha \in \mathfrak{h}^* \mid \text{root} \} = \Delta^+ \sqcup \Delta^-$$

$($ $\mathfrak{g}_\alpha \neq 0, \ \alpha \neq 0$

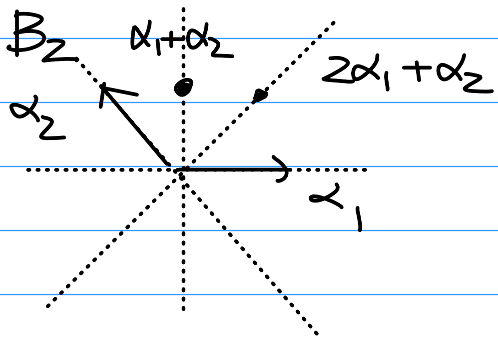
$W = \langle s_i \rangle$: Weyl group $\subset GL(\mathfrak{h}^*)$

$$s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$$

A_2



B_2



Weyl chamber: a connected component of $\mathfrak{h}^*_{\mathbb{R}} \setminus \bigcup (\alpha, \cdot) = 0$

<u>Fact</u> $\{\text{chambers}\} \curvearrowleft W$ simply transitive

<u>reduced expression</u>

$W \ni w = s_{i_1} \cdots s_{i_r}$

$$\min\{ r \mid w = s_{i_1} \cdots s_{i_r} \} = \ell(w)$$

length of w

If $\ell(w) = r$, $w = s_{i_1} \cdots s_{i_r}$, $\quad s_{i_1} \cdots s_{i_r}$ or $((i_1, \cdots, i_r))$

reduced expression of w

ex. $\ell(1) = 0$, $\ell(s_i) = 1$

- $\ell(w^{-1}) = \ell(w)$
- $w = s_{i_1} \cdots s_{i_r}$: reduced expr.
  $\Longrightarrow$ $s_{i_p} \cdots s_{i_q}$ $1 \leqq p \leqq q \leqq r$
  reduced expr.
- $\det(w) = (-1)^{\ell(w)}$
- $\ell(ws_i) = \ell(w) \pm 1$

$\mathfrak{g}$: finite type $\Longrightarrow$ W : finite group

$\exists w_0$ s.t. $\ell(w_0) =$ maximum

unique $C \longrightarrow -C$

chamber opposite

several reduced exprs

e.g. A_2 $w_0 = s_1 s_2 s_1$
$= s_2 s_1 s_2$

Recall
$W \curvearrowright \Delta$

<u>Lemma 1</u> $s_i(\Delta^+ \setminus \{\alpha_i\}) = \Delta^+ \setminus \{\alpha_i\}$

$\because$ $\Delta^+ \ni \alpha = \sum_j n_j \alpha_j$ $n_j \in \mathbb{Z}_{\geq 0}$ $\alpha \neq \alpha_i$

Assume

$\therefore \exists j_0 \neq i$ s.t. $n_{j_0} > 0$

$$s_i(\alpha) = \sum_{j \neq i} n_j \alpha_j + (n_i - \langle \alpha, h_i \rangle)\alpha_i \in \Delta^+ \quad //$$

Since $n_{j_0} > 0$

Lemma 2. $w = s_{i_1}\cdots s_{i_r}$ reduced expression

$$\Longrightarrow s_{i_1}\cdots s_{i_{r-1}}(\alpha_{i_r}) \in \Delta^+$$

∵ Consider $\alpha_{i_r}, s_{i_{r-1}}(\alpha_{i_r}), \dots, s_{i_1}\cdots s_{i_{r-1}}(\alpha_{i_r})$.

Suppose $\underbrace{s_{i_{\ell+1}}\cdots(\alpha_{i_r})}_{=\beta} \in \Delta^+$, but $s_{i_\ell}\cdots s_{i_{r-1}}(\alpha_{i_r}) \in \Delta^-$

$$\begin{cases} \beta \in \Delta^+ \\ s_{i_\ell}(\beta) \in \Delta^- \end{cases} \quad \therefore\ \beta = \alpha_{i_\ell} \text{ by Lemma 1}$$

$$\therefore\ s_{i_{\ell+1}}\cdots(\alpha_{i_r}) = \alpha_{i_\ell}$$

$$s_{i_\ell} = (s_{i_{\ell+1}}\cdots\ )(s_{i_r})(s_{i_{\ell+1}}\cdots\ )^{-1}$$

$$\therefore\ s_{i_{\ell+1}}\cdots s_{i_r} = s_{i_\ell}\cdots s_{i_{r-1}}$$

$$w = s_{i_1}\cdots s_{i_r} = s_{i_1}\cdots \underline{s_{i_\ell}}\ \underbrace{s_{i_{\ell+1}}\cdots s_{i_r}}_{=\ \underline{s_{i_\ell}}\cdots s_{i_{r-1}}}$$

1

contradicts with $w = s_{i_1}\cdots s_{i_r}$ red. expr. //

Prop. $\ell(w) = |\underbrace{\Delta^+ \cap w(\Delta^-)}_{=\{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1}\cdots s_{i_{r-1}}(\alpha_{i_r})\}}|$

($w = s_{i_1}\cdots s_{i_r}$ red. expr.)

∵ induction

- $\ell(w)=0 \iff w=1 \iff \Delta^+ \cap w\Delta^- = \phi$
- $w = s_{i_1}\cdots s_{i_r}$, $y = s_{i_1}\cdots s_{i_{r-1}}$ $\quad w = y s_{i_r}$

$$\Delta^+ \cap w(\Delta^-) = \Delta^+ \cap y \underbrace{s_{i_r}\Delta^-}_{\overset{\text{lemma 1}}{=}\ (\Delta^- \setminus \{-\alpha_{i_r}\}) \sqcup \{\alpha_{i_r}\}}$$

$$y\alpha_{i_r} = w(-\alpha_{i_r})$$
$$\| $$
$$s_{i_1}\cdots s_{i_{r-1}}(\alpha_{i_r}) \in \Delta^+ \quad \text{lemma 2}$$

$$= \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1}\cdots s_{i_{r-2}}(\alpha_{i_{r-1}})\} \sqcup \{y\alpha_{i_r}\}$$

↑ induction hypothesis

//

w_0 longest element
$= s_{i_1}\cdots s_{i_\nu}$ reduced expr.

$$\nu = \ell(w_0) = |\Delta^+ \cap w_0(\Delta^-)| = |\Delta^+|$$
$$(\therefore\ \ell(w_0) = |\Delta^+|)$$

e.g. A_2 $\ell(w_0)=3$
$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$

Remark (important later !)

- Reduced expr. of $w_0 \Longrightarrow$ total order on Δ^+

$$\Delta^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots \}$$

- total order depends on the choice of a reduced expr.

e.g. A_2 : $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$

total order: $\alpha_1,\ \underbrace{s_1(\alpha_2)}_{=\alpha_1+\alpha_2},\ \underbrace{s_1 s_2(\alpha_1)}_{=\alpha_2}$ $\qquad$ $\alpha_2,\ \alpha_1+\alpha_2,\ \alpha_1$

<u>Exercise</u>

Fix a red. expr of w_0

$\alpha^a :=$ ath positive root $\qquad 1 \leqq a < b \leqq \nu$

$\alpha^b =$ bth "

If $\alpha^a + \alpha^b$ is a positive root, then it appears between a & b.

(convex order)

PBW base of $U(\mathfrak{g})$ (prototype of PBW base of $U_q(\mathfrak{g})$)

finite type

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \Rightarrow U(\mathfrak{g}) = U(\mathfrak{n}^+) \otimes \underbrace{U(\mathfrak{h})}_{= S(\mathfrak{h})} \otimes U(\mathfrak{n}^-)$$

symmetric alg

$$\mathfrak{n} = \mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

Choose a red expr. w_0, and hence total order on Δ^+

$\{\alpha^1, \alpha^2, \cdots, \alpha^\nu\}$

Th (PBW Poincaré–Birkhoff–Witt theorem)

cf. [谷 1.24]

Choose a total order of Δ^+.

Choose $0 \neq e^{\alpha^a} \in \mathfrak{g}_{\alpha^a}$ for each $a = 1, \dots, \nu$

Then $(e^{\alpha^1})^{m_1} (e^{\alpha^2})^{m_2} \cdots (e^{\alpha^\nu})^{m_\nu} \in U(\mathfrak{n})$

$m_1, m_2, \dots, m_\nu \in \mathbb{Z}_{\geq 0}$

form a base of $U(\mathfrak{n})$ (as a vector space)

Exercise Give a proof of this theorem.

- PBW base depends on the total order.
- PBW base of QUE $U_q(\mathfrak{g})$

$\longrightarrow$ prototype PBW base of $U(\mathfrak{g})$

even worse

- $\times$ root subspace

analog of e^{α^a} is defined depending on the choice of reduced expr.

$$
\begin{aligned}
w_0 &= \cdots \\
&= \cdots
\end{aligned}
\quad \updownarrow \text{diff. red. expr} \quad \rightsquigarrow \quad e^{\alpha^a} \,/\, e'^{\alpha^a}
$$

$e'^{\alpha^a} \notin \mathbb{Q}(q)\, e^{\alpha^a}$

- total order depends on the red. expr.

<u>canonical base</u> FIXES these dependences

$w_0 = \cdots \longrightarrow$ PBW base $\longrightarrow$ canonical base

$\cdots \longrightarrow$ PBW base' $\nearrow$ same

different

independent of the reduced expression

canonical base

— useful for representation theory of $\mathfrak{g}$ & $U_q(\mathfrak{g})$

— useful for representation theory of <u>other</u> algebras

- LLT conj. Ariki

affine Hecke alg.
cyclotomic Hecke alg.

- quiver Hecke algebra

"std" representations / simple irreducible representations

elementary explicit construction ... difficult

PBW base ~~vs.~~ canonical base

transition matrix of PBW : canonical

= composition multiplicity of std rep : simple rep

cf Kazhdan-Lusztig conjecture

rep. th. of $\mathfrak{g}$ (∞-dim'l) Verma module : simple module

comp. mult. of

= transition matrix of "std" base : "canonical" base

Hecke alg.

q-deformation of $\mathbb{C}[W]$ ⇝ $\mathcal{H}_q$

Def. quantized universal enveloping algebra (QUE) $U_q(\mathfrak{g}) \equiv U_q$ is a $\mathbb{Q}(q)$-algebra defined by

generators : $e_i, f_i, t_i^{\pm}$ $(i \in I)$

$$t_i \cdot t_i^{-1} = 1 = t_i^{-1} t_i$$

$$t_i t_j = t_j t_i \qquad t_i = \text{"}q^{d_i h_i}\text{"}$$

$d_i a_{ij} = d_j a_{ji}$

$$t_i e_j t_i^{-1} = q^{d_i a_{ij}} e_j$$

$$t_i f_j t_i^{-1} = q^{-d_i a_{ij}} f_j$$

$q_i = q^{d_i}$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$$

quantum Serre relation

(Serre rel. $(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j)=0$)

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \underbrace{\begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i}}_{q\text{-binomial coefficient}} e_i^k e_j e_i^{1-a_{ij}-k} = 0$$

- $e \longleftrightarrow f$

coproduct $\quad \Delta t_i^{\pm} = t_i^{\pm} \otimes t_i^{\pm}$

$$\Delta e_i = e_i \otimes t_i^{-1} + 1 \otimes e_i$$
$$\Delta f_i = f_i \otimes 1 + t_i \otimes f_i$$

$$\varepsilon(t_i^{\pm}) = 1, \quad \varepsilon(e_i) = 0 = \varepsilon(f_i)$$

$$S(t_i^{\pm}) = t_i^{\mp}, \quad S(e_i) = -e_i t_i$$
$$S(f_i) = -t_i^{-1} f_i$$

$U_q(\mathfrak{g})$ is a Hopf algebra.

(Δ is compatible with the defining relations of $U_q(\mathfrak{g})$)

Exercise

Note $\quad U_{q_i}(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{g})$

$\quad \| $

$\langle e_i, f_i, t_i^{\pm} \rangle$

- alg. hom.
- Δ is also compatible

Def. $\overline{}$ bar involution (— used in the def. of canonical base in a crucial way!)

$\overline{} : U_q(\mathfrak{g}) \circlearrowleft$

$\overline{\overline{}} = id$

$\mathbb{Q}$-alg. hom, but not $\mathbb{Q}(q)$-alg. hom.

- $\overline{q} = q^{-1}$, $\overline{t_i} = t_i^{-1}$, $\overline{e_i} = e_i$, $\overline{f_i} = f_i$

This preserves the defining relations.

- $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ $\quad \overline{[n]_q} = [n]_q$

$\rightsquigarrow \overline{\begin{bmatrix} n \\ k \end{bmatrix}_q} = \begin{bmatrix} n \\ k \end{bmatrix}_q$

ok q-Serre rel

$U_q^+ = \langle e_i \rangle$, $U_q^0 = \langle t_i^{\pm} \rangle$, $U_q^- = \langle f_i \rangle$

Fact. triangular decomp.

$$U_q \cong U_q^+ \otimes U_q^0 \otimes U_q^-$$

weight subspace

$\xi \in Q = \bigoplus \mathbb{Z}\alpha_i$ (root lattice)

$(U_q)_\xi \overset{\text{def.}}{=} \{ x \in U_q \mid t_i x t_i^{-1} = q^{\langle d_i h_i, \xi \rangle} x \ \forall i \}$

$(U_q^{\pm})_\xi$: similar $\qquad$ e.g. $e_i \in (U_q)_{\alpha_i}$

○ representation theory (finite dim'l repr.)

Def. A U_q-module M has a (type I) weight space decomposition

$\underset{\text{def.}}{\iff}$ $M = \bigoplus M_\lambda$

$\lambda \in \mathrm{Hom}(Q^*, \mathbb{Z})$ weight lattice

$Q^* = \bigoplus \mathbb{Z} h_i$

$M_\lambda = \{ m \in M \mid t_i m = q^{\overbrace{\langle d_i h_i, \lambda \rangle}^{\in \mathbb{Z}}} m \}$ $\forall i$

λ s.t. $M_\lambda \neq 0$ is called a <u>weight</u> of M

Def. $0 \neq M$ (has weight space decomp) is a highest weight module of h.w. $= \lambda$

$\iff \exists m \in M_\lambda$ s.t. $e_i m = 0 \quad \forall i$

$U_q^- m = M$

$\Longrightarrow$ any wt μ of M satisfies $\mu \leq \lambda$

$\lambda - \mu \in Q_+ = \sum_i \mathbb{Z}_{\geq 0} \alpha_i$ (dominance order)

Verma module $M(\lambda)$: universal highest wt module

$$M(\lambda) = U_{\mathfrak{g}} / J_\lambda$$

J_λ = left ideal generated by
$e_i,\ t_i - \xi^{\langle d_i \alpha_i, \lambda \rangle} \mathrm{id}$

$$= \sum U_{\mathfrak{g}} e_i + \sum U_{\mathfrak{g}} (t_i - \cdots)$$

$$\begin{array}{ccc} M(\lambda) & \twoheadrightarrow & M \\ \Psi & & \Psi \\ 1 \bmod J_\lambda & \mapsto & m \\ \| & & \\ v_\lambda \text{ or } m_\lambda & & \end{array}$$

any h.w. module with h.w $= \lambda$