Review

$\mathfrak{g}$: Kac Moody Lie alg.

$U_q \equiv U_q(\mathfrak{g})$: QUE $= \langle e_i, f_i, t_i^{\pm} \rangle / \text{rel}$

$\uparrow$ Hopf algebra

M : U_q - module has a weight space decomposition

$$M = \bigoplus_{\lambda \in P = \mathrm{Hom}(\underset{\bigoplus \mathbb{Z} h_i}{\underset{|}{Q}}, \mathbb{Z})} M_\lambda \qquad M_\lambda = \{ m \in M \mid \underset{q^{\langle d_i h_i, \lambda \rangle} m}{\underset{\|}{t_i m}} \}$$

λ s.t. $M_\lambda \neq 0$ weight of M

highest weight module $M \ni m \leftarrow \mathrm{wt} = \lambda$

$$\begin{cases} e_i m = 0 \\ M = U_q \cdot m \end{cases}$$

universal highest weight module $=$ Verma module $M(\lambda)$

$= U_q / J_\lambda$

$J_\lambda =$ left ideal generated by e_i, $t_i - q^{\langle d_i h_i, \lambda \rangle}$

Def. M : integrable module

$\overset{}{\underset{\text{def.}}{\iff}}$ e_i, f_i locally nilpotent for $\forall i$

i.e. $\forall m \in M$ $\exists N >> 0$

$e_i^N m = 0 = f_i^N m$ $\forall i$

Fact (1) Any integrable module is a direct sum of simple integrable modules

(complete reducibility)

(2) irreducible integrable module

$\Rightarrow$ highest weight

$\therefore$ quotient of Verma module

$$V_{\mathfrak{g}}(\lambda) \twoheadleftarrow M(\lambda)$$

(integrable $\iff$ λ is dominant

$$P_+ = \{\lambda \in P \mid \langle \lambda, h_i \rangle \geqq 0 \ \forall i\}$$

Moreover $V_{\mathfrak{g}}(\lambda)$ is $M(\lambda) \Big/ \sum_i U_{\mathfrak{g}} \underbrace{f_i^{\langle \lambda, h_i \rangle + 1}}_{}$

explained later

($\Rightarrow$ dim of wt spaces for $V_{\mathfrak{g}}(\lambda)$

= same as $V(\lambda)$ for KM.

(3) $\mathfrak{g}$: finite type $\Rightarrow$ $V_{\mathfrak{g}}(\lambda)$: finite dim'l rep

§ specialization at $q=\underline{1}$

(<u>cheap</u> version of specialization)

$$\mathbb{F} = \mathbb{Q}(q) \supset \mathbf{A}_1 = \{ f \in \mathbb{Q}(q) \mid f \text{ is regular at } q=1 \}$$

$$\not\ni \frac{1}{1-q}$$

(local ring

max. ideal $(q-1)\mathbf{A}_1$

${}_{\mathbf{A}_1}U_q :=$ $\mathbf{A}_1$-subalgebra generated by

$$e_i,\ f_i,\ \underset{\underset{q^{d_i h_i}}{\shortparallel}}{t_i^{\pm}},\ \frac{t_i^{\pm}-1}{q_i^{\pm}-1}$$

$\cup$

$\underbrace{{}_{\mathbf{A}_1}U_q^{\pm}}$, ${}_{\mathbf{A}_1}U_q^{0}$

$\cup$

$({}_{\mathbf{A}_1}U_q^{\pm})_{\xi}$ weight space

(${}_{\mathbf{A}_1}U_q \otimes_{\mathbf{A}_1} \mathbb{Q}$ (specialization)

$$\begin{tikzcd} \mathbf{A}_1 \arrow[r] & \mathbb{Q} \\ f(q) \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & f(1) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$
)

<u>Th</u>. (1) ${}_{\mathbf{A}_1}U_q \cong {}_{\mathbf{A}_1}U_q^{+} \otimes {}_{\mathbf{A}_1}U_q^{0} \otimes {}_{\mathbf{A}_1}U_q^{-}$ $\left(\otimes_{\mathbf{A}_1} = \otimes\right)$

(2) $({}_{\mathbf{A}_1}U_q^{\pm})_{\xi}$ is a <u>free</u> $\mathbf{A}_1$-module of finite rank

(3) ${}_{\mathbf{A}_1}U_q \otimes_{\mathbf{A}_1} \mathbb{Q} \cong U(\mathfrak{g})$

universal enveloping algebra of KM Lie alg.

(sketch of proof)

(1) same as the usual case

(2) finitely generated torsion free $\underset{\overset{\nearrow}{\text{PID}}}{A_1}$-module $\Rightarrow$ free

(3) $${}_{A_1}U_q \otimes_{A_1} \mathbb{Q} \ni \text{image of } e_i, f_i, \frac{\overset{\overset{q^{d_i h_i}}{\|}}{t_i}-1}{\underset{\underset{q^{d_i}}{\|}}{q_i}-1} \xrightarrow{} h_i$$

These satisfy the defining relation of KM Lie alg.

$$\Longrightarrow \exists\ U(\mathfrak{g}) \twoheadrightarrow {}_{A_1}U_q \otimes_{A_1} \mathbb{Q}$$

alg. hom

$$e_i, f_i, h_i \longmapsto \text{images}$$

Want to show this is an isom.

— enough to show

$$\begin{tikzcd} U(\mathfrak{n}^-) \arrow[r, two heads] & {}_{A_1}U_q^- \otimes_{A_1} \mathbb{Q} \\ \text{limit of } V(\lambda),\ \lambda \to \infty \arrow[u, "\wr"] & \text{limit of } V_q(\lambda),\ \lambda \to \infty \arrow[u, "\wr"] \end{tikzcd}$$

— enough to show for integrable highest wt module

$$V(\lambda) \xrightarrow{\ \star\ } \!\!\!\!\rightarrow \underbrace{{}_{A_1}V_q(\lambda)}_{= {}_{A_1}U_q^- \cdot v_\lambda} \otimes_{A_1} \mathbb{Q}$$

Fact. · $V(\lambda) = U(\mathfrak{n}^-) \Big/ \sum_i U(\mathfrak{n}^-) f_i^{\langle \lambda, h_i \rangle + 1}$

· irreducible

☆ must be an isom. by the irreducibility of $V(\lambda)$.

$$\Rightarrow {}_{\mathbb{A}_1}V_\xi(\lambda) \cong {}_{\mathbb{A}_1}U_\xi^- \Big/ \sum_i {}_{\mathbb{A}_1}U_\xi^- f_i^{\langle \lambda, h_i \rangle + 1}$$

We will write $V_\xi(\lambda)$ simply by $V(\lambda)$ hereafter.

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§ Braid group action on U_ξ

We follow

Saito: PBW basis of QUEs
Publ. of RIMS 30 (1994)

(Jantzen's textbook)

Lusztig's textbook ← computation is harder to follow

1st step (braid group) operator T on integrable rep. V of $U_\xi(\mathfrak{sl}_2)$

$V \ni v$ weight m

$$\underset{V \to V}{T}(v) := \exp_{q^{-1}}(q^{-1} e t^{-1}) \exp_{q^{-1}}(-f) \exp_{q^{-1}}(q e t) \times q^{\frac{m(m+1)}{2}} v$$

$$\exp_{q^{-1}}(X) = \sum_{k=0}^{\infty} q^{-\frac{k(k-1)}{2}} \frac{X^k}{[k]!}$$

(Recall $SL_2 \ni \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \exp(\quad)\exp(\quad)\exp(\quad)$

Exercise $(\exp_q X)^{-1} = \exp_{q^{-1}}(-X)$

(q-binomial thm)

Rem V: integrable $\Rightarrow$ finite sum well-defined

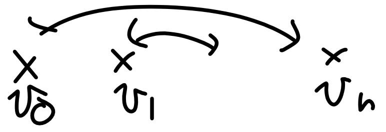
$e^{(a)} := \frac{e^a}{[a]!}$ q-divided power

Prop 1 (1) $T(v) = \sum_{\substack{a,b,c \geqq 0 \\ -a+b-c=m}} (-1)^b q^{b-ac} e^{(a)} f^{(b)} e^{(c)} v$

v : wt $= m$

(2) $V = V(n)$ irreducible f.d. rep. h.w $= n$

Let $v_i := f^{(i)} \underline{v_0}$ ← h.w $T(v_i) = (-1)^{n-i} q^{(n-i)(i+1)} v_{n-i}$



(Sketch of proof)

(1) enough to show for $V = V(n)$

(1). (2) $\leftarrow$ direct computation for $V(n)$ using q-binomial thm. etc

Exercise Fill the detail.

Prop 2 V : integrable rep $\ni v$

(1) $T(tv) = t^{-1}T(v)$

(2) $T(ev) = (-ft)T(v)$

(3) $T(fv) = (-t^{-1}e)T(v)$

(proof) (1) clear

(2) $T(ev_i) = [n+1-i]\,T(v_{i-1}) = \cdots\cdots$

$-ft\,T(v_i) = \cdots\cdots$ $\swarrow\!\!\nearrow$ same

(3)

//

M, N : integrable $\Longrightarrow$ $M\otimes N$ integrable

but T on $M\otimes N \neq T_{\text{on } M} \otimes T_{\text{on } N}$

$$L := \sum_{n\geq 0} q^{\frac{n(n-1)}{2}} \prod_{a=1}^{n} (q^{a}-q^{-a})\, e^{(n)}\otimes f^{(n)}$$

Prop 3. L is invertible

$$\underset{\sim}{L} \circ \underset{\overset{\uparrow}{M\otimes N}}{T}(x\otimes y) = \overset{M}{T}x \otimes \overset{N}{T}y \qquad x\in M,\ y\in N$$

(L is analog of R-matrix)

2nd Step T_i for integrable rep. of $U_q(\mathfrak{g})$.

V : integrable $U_q(\mathfrak{g})$-module

$$\begin{array}{l} U_q(\mathfrak{g}) \\ \quad \cup \\ \quad U_{q_i}(\mathfrak{sl}_2) = \langle e_i, f_i, t_i^{\pm} \rangle \end{array}$$

$V|_{U_{q_i}(\mathfrak{sl}_2)}$: integrable $U_{q_i}(\mathfrak{sl}_2)$-mod.

$\circlearrowleft$ step1 gives an operator

$$T = T_i$$

3rd Step operators acting on $U_q(\mathfrak{g})$

S, S' : operators acting on integrable rep. V

$\underset{\text{invertible}}{\overset{\uparrow}{S}}$

$$\underset{SS'S^{-1}}{\overset{\text{def.}\,\|}{Ad(S)S'}} : \text{operator acting on } V$$

Th (1) $\forall x \in U_q$ (considered as an operator acting on V)

Consider $Ad\,T_i(x) = T_i x T_i^{-1}$: operator on V.

This is represented by an element x' in U_q

i.e. $$T_i(x\underset{V}{\underset{\uparrow}{v}}) \underset{☆}{=} x' T_i(v) \quad \forall v \in \forall V$$

moreover x' is unique

(2) $\underset{\underset{U_q}{\cap}}{x} \longmapsto \underset{\underset{\hat{U}_q}{\cap}}{x'}$ is an algebra ism. — $\mathrm{Ad}\,T_i(x)$

(3) $\mathrm{Ad}\,T_i(x)$ is given on generators by

- $\underset{\underset{q^{d_j h_j}}{\|}}{t_j} \longmapsto t_i^{-a_{ij}} t_j$

$$q^{s_i(d_j h_j)} = q^{d_j h_j - \underbrace{\langle \alpha_i, d_j h_j \rangle}_{\underset{d_j a_{ji} = d_i a_{ij}}{\|}} h_i}$$

- $e_i \mapsto -f_i t_i$, $f_i \mapsto -t_i^{-1} e_i$

- $(i \neq j)$ $e_j \longmapsto \displaystyle\sum_{r=0}^{-a_{ij}} (-1)^r q_i^{-r} \overbrace{e_i^{(-a_{ij}-r)}} e_j \overset{\overset{\frac{e_i^r}{[r]!}}{\|}}{\overbrace{e_i^{(r)}}}$

$$f_j \longmapsto \sum_{r=0}^{-a_{ij}} (-1)^{-a_{ij}-r} q_i^{-a_{ij}-r} f_i^{(-a_{ij}-r)} f_j f_i^{(r)}$$

Rem. (3) can be used to define operator $\mathrm{Ad}(T_i)$ on U_q
(need to check its compatibility with the defining relations of U_q)

(sketch of the proof)

- uniqueness

If x' & x'' satisfy ☆

$$T_i(xv) = x' T_i(v) = x'' T_i(v)$$

$(x'-x'')$ acts on V by 0

(**Fact**. If $x \in U_q$ acts on all integrable modules by 0
$\Longrightarrow x=0$

$\therefore \ x'=x''$

(2) $\mathrm{Ad}T_i$ alg. isom $\longleftarrow$ follows from the uniqueness

(3) enough to show the existence of $\mathrm{Ad}T_i(x)$ for generators $x=t_j^{\pm}, e_j, f_j$

t_j $\qquad T_i : \underset{\substack{| \\ \text{weight}=\mu}}{V_\mu} \xrightarrow{\simeq} V_{s_i\mu}$

e_i, f_i : Prop2 ($T(ev) = \underbrace{(-ft)}_{\substack{\shortparallel \\ \mathrm{Ad}T(e)}} T(v)$

$e_j, f_j \xrightarrow{\mathrm{Ad}T_i} ?$

Recall adjoint action of $\mathfrak{g}$ on itself $\supset (\mathfrak{sl}_2)_i = \langle e_i, f_i, h_i \rangle$

$$\begin{tikzcd} 0 & f_j \arrow[l, bend right, "{0=[e_i,f_j]}"'] \arrow[r, bend left] & {[f_i,f_j]} \arrow[r, bend left] & \cdots \arrow[r, bend left] & (\mathrm{ad} f_i)^{-a_{ij}}(f_j) \arrow[r, bend left] & (\mathrm{ad} f_i)^{1-a_{ij}}(f_j)=0 \end{tikzcd}$$

Serre rel.

Weyl reflection

$$\begin{tikzcd} \bullet \arrow[rrr, bend right] & \bullet \arrow[r, bend right] & \cdots & \bullet \arrow[lll, bend right] \end{tikzcd}$$

"$\mathrm{Ad}T_i(f_j)$"

$$\overset{\frac{e_i^{-a_{ij}-r}}{[-a_{ij}-r]!}\ \parallel}{\overbrace{e_i^{(-a_{ij}-r)}}}\ e_j\ \overset{\parallel\ \frac{e_i^{r}}{[r]!}}{\overbrace{e_i^{(r)}}} \qquad \begin{bmatrix} -a_{ij} \\ r \end{bmatrix}_q \qquad \binom{-a_{ij}}{r} = \underline{\frac{(-a_{ij})!}{r!\,(-a_{ij}-r)!}}$$

(

similar to the expr. appearing in q-Serre rel,
but for $\underline{-a_{ij}}$ instead of $\underline{1-a_{ij}}$

We should consider an analog of adjoint rep. of $\mathfrak{g}$ on itself for $U_q(\mathfrak{g})$

$$\left(\begin{array}{l} A \underset{x,y}{\ni} : \text{Hopf alg.}/k \qquad \mathrm{ad}(x)(y) \overset{\text{def.}}{=} \sum x_i\, y\, S(x_i') \\ \qquad\qquad\qquad\qquad\qquad\qquad \Delta x = \sum_i x_i \otimes x_i' \\ \qquad\qquad \mathrm{ad} : A \longrightarrow \mathrm{End}_k(A) \quad \text{alg. hom.} \end{array}\right.$$

$$\left(\begin{array}{l} \mathrm{ad}(f_i)(x) = f_i x - t_i x t_i^{-1} f_i \\ \mathrm{ad}(e_i)(x) = e_i x t_i - x e_i t_i \end{array}\right.$$

variant $\mathrm{ad}^*(x)(y) \overset{\text{def.}}{=} \left(\mathrm{ad}(x)(y^*)\right)^* \swarrow$ anti alg. isom. defined last week

Lemma $\displaystyle ad(f_i^{(m)})(x) = \sum_{r+s=m} (-1)^r q_i^{r(1-m)} f_i^{(s)} t_i^r x t_i^{-r} f_i^{(r)}$

$$ad^*(f_i^{(m)})(x) = \sum_{r+s=m} (-1)^r q_i^{r(1-m)} f_i^{(r)} t_i^r x t_i^{-r} f_i^{(s)}$$

☺ by induction

q-Serre relation $\iff ad(f_i^{(1-a_{ij})})(f_j) = 0$

$\iff ad^*(f_i^{(1-a_{ij})})(f_j) = 0$

The above formula of $Ad(T_i)(f_j) = \boxed{\ast}$

$ad^*(f_i^{(-a_{ij})})(f_j) = \nearrow$

Lemma 4 $V \overset{\text{def.}}{=} \displaystyle\bigoplus_{k=0}^{-a_{ij}} \mathbb{Q}(q)\, ad^*(f_i^{(k)})(f_j)$
$i \neq j$

is a $U_{q_i}(\mathfrak{sl}_2)$-module via ad^*-action
$\qquad\qquad \| \langle e_i, f_i, t_i^{\pm} \rangle$

$f_j \cdots$ h.w. vector $wt = -a_{ij}$

😮 $ad^*(e_i)(f_j) = \cdots = 0$ //

Rest of the proof is basically computation (no conceptual additional input)

Exercise Check the remaining part.