Recall

$T_i : V \circlearrowleft$

integrable repr.

$\mathrm{Ad}T_i : U_q \circlearrowleft$ s.t. $T_i(\underset{\underset{U_q}{\cap}}{x}\underset{\underset{V}{\cap}}{v}) = \mathrm{Ad}T_i(x)\, T_i(v)$

on generators : $f_i \mapsto -t_i^{-1} e_i$

$t_j \mapsto t_i^{-a_{ij}} t_j$

$$f_j \mapsto \mathrm{ad}^*(f_i^{(-a_{ij})})(f_j)$$
$$\| $$
$$\sum_{r+s=-a_{ij}} (-1)^s q_i^s f_i^{(s)} f_j f_i^{(r)}$$

<u>**Key Lemma 4**</u>
$i \neq j$

$$V = \bigoplus_{k=0}^{-a_{ij}} \mathbb{Q}(q)\, \mathrm{ad}^*(f_i^{(k)})(f_j)$$

is a $U_{q_i}(\mathfrak{sl}_2) = \langle e_i, f_i, t_i \rangle$-module via ad^*-action

h.w. vector $= f_j$, wt $= -a_{ij}$

<u>**Lemma 5**</u> $\mathrm{Ad}T_i(\mathrm{ad}(f_i)(x)) = \mathrm{ad}^*(e_i)\, \mathrm{Ad}T_i(x)$

$\because$ LHS $= \mathrm{Ad}T_i(f_i x - t_i x t_i^{-1} f_i)$

$= (-t_i^{-1} e_i)\, \mathrm{Ad}T_i(x) + t_i^{-1} \mathrm{Ad}T_i(x)\, t_i t_o^{-1} e_i$

$=$ RHS //

$$\mathrm{ad}^*(x)(y) = (\mathrm{ad}(x)(y^*))^*$$

Prop 6. $\mathrm{Ad}\,T_i\,(\mathrm{ad}(f_i^{(m)})(f_j)) = \mathrm{ad}^*(f_i^{(-a_{ij}-m)})(f_j)$

∵ $\mathrm{Ad}\,T_i\,(\mathrm{ad}\,f_i^{(m)})(f_j)) = \mathrm{ad}^*(e_i^{(m)})\,\mathrm{Ad}\,T_i(f_j)$

Lemma 5 $\quad = \mathrm{ad}^*(e_i^{(m)})\,\mathrm{ad}^*(f_i^{(-a_{ij})})(f_j)$

$$\underset{\nearrow \text{ Lemma 4}}{=} \mathrm{RHS} \quad //$$

Cor.7 $\mathrm{Ad}\,T_i^{-1}(f_j) = \underset{\uparrow \text{ not } \mathrm{ad}^*}{\mathrm{ad}}(f_i^{(-a_{ij})})(f_j)$

∵ $m = -a_{ij}$ in Prop.6 //

Note $\mathrm{Ad}\,T_i \neq \mathrm{Ad}\,T_i^{-1}$ ⇝ difference between braid group & Weyl group

Cor 8. $\mathrm{Ad}\,T_i^{-1} = *\,\mathrm{Ad}\,T_i\,*$

∵ enough to check on generators

Cor 7 $\Longrightarrow$ OK for f_j (and e_j)

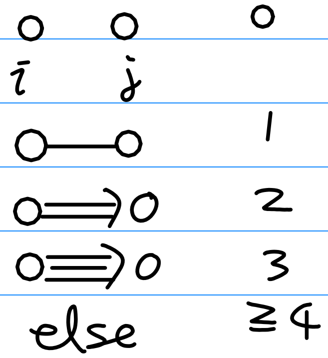
$\mathrm{Ad}\,T_i(f_i) = -t_i' e_i$ ⇝ OK for f_i, e_i

t_i : easy //

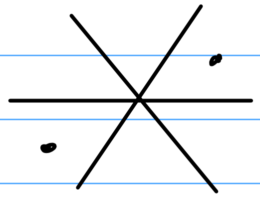
$i \neq j$ $\quad a_{ij}a_{ji} = 0, 1, 2, 3, \geq 4$

$\rightsquigarrow h(i,j) \overset{\text{def}}{=} 2, 3, 4, 6, \infty$

$=$ length of w_0 for rk 2
Lie alg. gen. by
$e_i, f_i, e_j, f_j, h_i, h_j$



e.g.
1



$h(i,j) = 3$

Th 1 Assume $h(i,j) < \infty$

$$\mathrm{Ad}(\underbrace{\cdots T_i T_j}_{h(i,j)-1})(f_i) = \begin{cases} f_j & h(i,j) : \text{odd} \\ f_i & \quad\text{even} \end{cases}$$

(analog of $\tilde{s}_i \tilde{s}_j (f_i) = f_j$

$s_i s_j (\alpha_i)$

(proof) 1° $a_{ij} = a_{ji} = 0$

$\mathrm{Ad}\,T_j(f_i) = f_i \qquad \mathrm{Ad}\,T_i(f_j) = f_j$

2° Assume $a_{ji} = -1$

$$\mathrm{ad}(f_i)(f_j) = f_i f_j - \underbrace{q_i^{-a_{ij}}}_{\substack{\| \\ q^{-d_i a_{ij}} = q^{-d_j a_{ji}}}} f_j f_i$$

$$= f_i f_j - q_j f_j f_i = \mathrm{ad}^*(f_j)(f_i)$$

Similarly $\mathrm{ad}^*(f_i)(f_j) = \mathrm{ad}(f_j)(f_i)$

3°. $a_{ij} = -1 \quad (= a_{ji})$

$$\mathrm{Ad}\,{}'T_j(f_i) = \mathrm{ad}^*(f_j)(f_i) = \mathrm{ad}(f_i)(f_j)$$

$$\underset{\substack{\uparrow \\ \text{Cor.7}}}{=} \mathrm{Ad}\,{}'T_i^{-1}(f_j)$$

$$\therefore \mathrm{Ad}\,{}'T_i\,\mathrm{Ad}\,{}'T_j(f_i) = f_j$$

4.° $a_{ij} = -2$

$$\mathrm{Ad}\,{}'T_i\,{}'T_j(f_i) = \mathrm{Ad}\,{}'T_i\;\mathrm{ad}^*(f_j)(f_i)$$

$$= \mathrm{Ad}\,{}'T_i\;\mathrm{ad}(f_j)(f_i)$$

$$\underset{\text{Prop.6}}{=} \mathrm{ad}^*(f_i)(f_j) = \mathrm{ad}f_j(f_i)$$

$$= \mathrm{Ad}\,{}'T_j^{-1}(f_i)$$

$$\mathrm{Ad}\,{}'T_j\,{}'T_i(f_j) = \mathrm{Ad}\,{}'T_j\;\underset{\displaystyle \frac{1}{[2]_{f_i}}\Big(\mathrm{ad}f_j\,\mathrm{ad}f_i(f_i) - f_i\,\mathrm{ad}f_j(f_i)\Big)}{\underset{\|}{\underline{\mathrm{ad}^*(f_i^{(2)})(f_j)}}}$$

$$= \cdots = \mathrm{ad}f_i^{(2)}(f_j) = \mathrm{Ad}\,{}'T_i^{-1}(f_j)$$

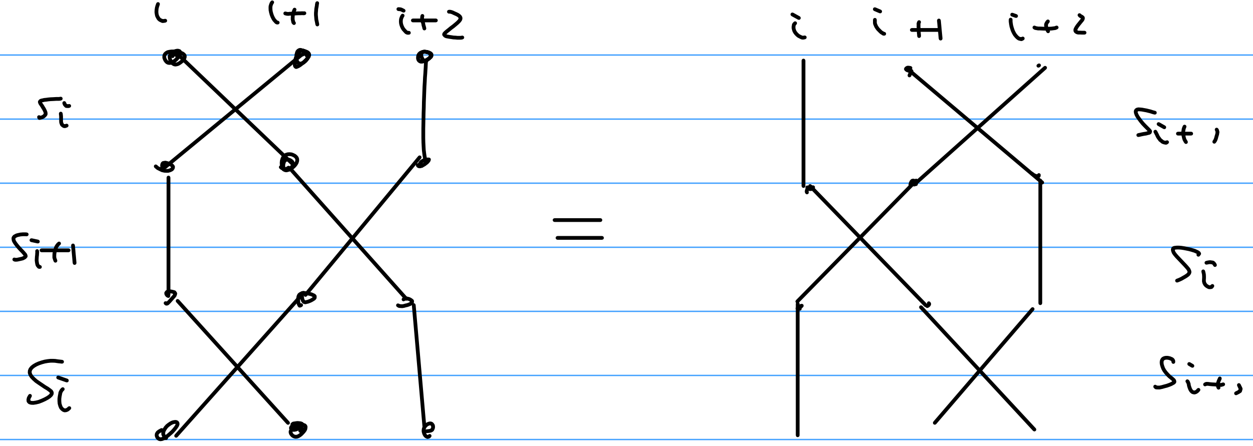
5.° $a_{ij} = -3$ <u>Exercise</u>

//

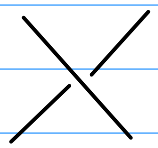
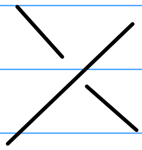
§ Braid group and Weyl group

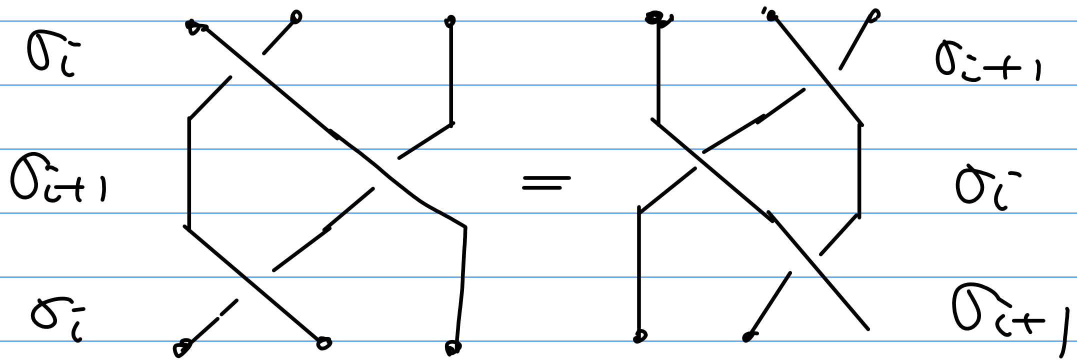
type A_{n-1} $W = \mathfrak{S}_n = \langle s_1, \cdots, s_{n-1} \rangle$

$$\begin{cases} s_i^2 = 1 \\ s_i s_j = s_j s_i \quad \text{if } |i-j|>1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \end{cases}$$



braid group B_n drop the 1st relation $s_i^2 = 1$

$\sigma_i =$  $\sigma_i^{-1} =$  ← 2d projection of 3d braid



<u>Th</u>. Weyl group has a presentation

- generators s_i $(i \in I)$

- relations s_i^2

$$(s_is_j)^{m(i,j)} = 1$$

$m(i,j) =$
2 $\quad \underbrace{s_is_j}\underbrace{s_is_j} = 1$
3 $\quad \underbrace{s_is_js_i}\underbrace{s_js_is_j} = 1$

Def. $B_W \overset{\text{def.}}{=} \langle \sigma_i \mid \underbrace{\sigma_i\sigma_j\cdots}_{m(i,j)\text{ terms}} = \sigma_j\sigma_i\cdots \rangle$ $\forall i \neq j$

braid group associated with the Weyl group W

We prepare several propositions to prove th.

Prop 1 (1) $w(\alpha_i) \in \Delta^- \Longrightarrow \exists$ reduced expr. of w ending with s_i

(2) $w = s_{i_1}\cdots s_{i_k} = s_{j_1}\cdots s_{j_k}$ (two reduced expr. of w)

$\Longrightarrow 1 \leqq {}^{\exists}a \leqq k$

s.t. $s_{i_1}\cdots \widehat{s_{i_a}}\cdots s_{i_k} = s_{j_1}\cdots s_{j_{k-1}}\widehat{s_{j_k}}$

(∵) (1) $\Delta^+ \cap w(\Delta^-) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots\}$ ← Prop. in 10??

$\Delta^+ \ni \alpha_i$ $\exists$

$\alpha_i = s_{i_k}\cdots s_{i_{a+1}}(\alpha_{i_a})$ ← Need to change the order

$s_i = (s_{i_k}\cdots s_{i_{a+1}})s_{i_a}(\;\;"\;\;)^{-1}$

$ws_i = s_{i_1}\cdots s_{i_{a-1}}\underbrace{s_{i_a}\cdots s_{i_{k-1}}s_{i_k}\cdot s_i}_{}$

$$= s_{i_1} \cdots s_{i_{a-1}} s_{i_{a+1}} \cdots s_{i_k}$$

(2) $w(\alpha_{j_k}) \in \Delta^-$

$$\underset{\text{above}}{w s_{j_k} =} \; s_{i_1} \cdots \hat{s}_{i_a} \cdots s_{i_k} \quad //$$

Prop. 2 $w = s_{i_1} \cdots s_{i_k}$ reduced expr.

$\Longrightarrow$ $\tilde{w} := \sigma_{i_1} \cdots \sigma_{i_k} \in B_W$

this is independent of the choice of the reduced expr. of w.

i.e.,

$$\begin{tikzcd} B_W \arrow[r, two heads] & W \\ \hat{w} \arrow[u, phantom, "\in" description] & w \arrow[l, mapsto] \arrow[u, phantom, "\in" description] \end{tikzcd}$$

well-defined

$\tilde{w} \cdot \tilde{w'} = \widetilde{ww'}$

if $\ell(ww') = \ell(w) + \ell(w')$

(proof) Suppose $s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k} = w$ $k = \ell(w)$

induction on k reduced expr.

- $k = 1 \Rightarrow$ clear
- $s_{i_1} \cdots \hat{s}_{i_a} \cdots s_{i_k} \underset{(\star)}{=} s_{j_1} \cdots s_{j_{k-1}}$ by Prop1

Case 1° $a > 1$

$$w = s_{j_1} \cdots s_{j_k} = s_{i_1} \cdots \hat{s}_{i_a} \cdots s_{i_k} s_{j_k}$$

$$\therefore s_{i_{a+1}} \cdots s_{i_k} s_{j_k} = s_{i_{a-1}} \cdots s_{i_1} w$$

$$= \underbrace{s_{i_a} \cdots s_{i_k}}_{\text{length} < k}$$

by the assump.

induction hypo. $\Rightarrow$ $\sigma_{i_a} \cdots \sigma_{i_k} = \sigma_{i_{a+1}} \cdots \sigma_{i_k} \sigma_{j_k}$

(⊛) + induct. hypo $\Rightarrow$ $\sigma_{i_1} \cdots \hat{\sigma}_{i_a} \cdots \sigma_{i_k} = \sigma_{j_1} \cdots \sigma_{j_k}$

$$\Longrightarrow \sigma_{j_1} \cdots \sigma_{j_k} = \sigma_{i_1} \cdots \sigma_{i_k}$$

OK

<u>case 2°</u> $a = 1$

i.e. $s_{i_2} \cdots s_{i_k} = s_{j_1} \cdots s_{j_{k-1}}$

$$w = s_{j_1} \cdots s_{j_{k-1}} s_{j_k} = s_{i_2} \cdots s_{i_k} s_{j_k}$$

Apply the same argument for $\vec{i} \longleftrightarrow \vec{j}$.

If case 1° $\longrightarrow$ OK by induction

Case 2° also holds for $\vec{i} \longleftrightarrow \vec{j}$

$$\begin{aligned} w &= s_{j_2} \cdots \underline{s_{j_k}}\, \underline{s_{i_k}} \\ &= s_{i_2} \cdots \underline{s_{i_k}}\, \underline{s_{j_k}} \end{aligned}$$

We repeat the above argument to $\vec{i}'' = (i_2 \cdots i_k j_k)$, $\vec{j}' = (j_2 \cdots j_k i_k)$

$\longrightarrow$ If case 1°, OK by induction

case 2° $\Rightarrow$ $w = s_{j_3} \cdots \underline{s_{j_k}}\, \underline{s_{i_k}}\, \underline{s_{j_k}}$

$= s_{i_3} \cdots \underline{s_{i_k}}\, \underline{s_{j_k}}\, \underline{s_{i_k}}$

$\rightsquigarrow$ braid relation OK induction hypo. //

Use

(proof of Thm)

$$\begin{tikzcd} B_W/\sigma_i^2=1 \arrow[r, two heads, "\overset{?}{\cong}", "\pi"'] \arrow[d, phantom, sloped, "\ni"] & W \arrow[d, phantom, sloped, "\ni"] \\ \mu(w)=\sigma_{i_1}\cdots\sigma_{i_k} & w=s_{i_1}\cdots s_{i_k} \arrow[l, mapsto, "\mu"] \end{tikzcd}$$

as a map between sets
well-defined

$\pi\circ\mu=\mathrm{id}$ OK

Show $\mu\circ\underbrace{\pi(\sigma_{i_1}\cdots\sigma_{i_k}}_{\parallel \atop s_{i_1}\cdots s_{i_k}})=\sigma_{i_1}\cdots\sigma_{i_k}$

by induction on k

If $s_{i_1}\cdots s_{i_k}$ is a reduced expr.
$\Rightarrow$ OK

If not reduced,

$\exists a$ $s_{i_1}\cdots s_{i_{a-1}}$: reduced
$s_{i_1}\cdots s_{i_a}$: not reduced

$$s_{i_1}\cdots s_{i_{a-1}}=s_{j_1}\cdots s_{j_{a-2}}s_{i_a}$$

$$\therefore\ \sigma_{i_1}\cdots\sigma_{i_{a-1}}=\sigma_{j_1}\cdots\sigma_{j_{a-2}}\sigma_{i_a}$$

$$\sigma_{i_1}\cdots\sigma_{i_{a-1}}\sigma_{i_a}\cdots\sigma_{i_\ell}=\sigma_{j_1}\cdots\sigma_{j_{a-2}}\underbrace{\sigma_{i_a}^2}_{\parallel \atop 1}\cdots$$

$\longrightarrow$ length is shorten
OK by induc. hypo. //

References　堀田　線型代数群の基礎
付録 B
Humphreys　Reflection Groups
and Coxeter groups

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5 minutes break

Return back to T_i

<u>Th 2</u> (1) T_i satisfies $\underbrace{T_i T_j \cdots}_{m(i,j) \text{ terms}} = T_j T_i \cdots$

the braid group relation

(2) $\mathrm{Ad}\, T_i$ also

(proof) enough to show (1)

(1) $a_{ij} = 0$ clear

$a_{ij} = -1 = a_{ji}$　　$T_i T_j T_i = T_j T_i T_j$

Want to show

$\Downarrow$

$(q_i = q_j = q^{d_i - d_j})$

$$\Longleftrightarrow T_i \underbrace{T_j T_i T_j^{-1}}_{\mathrm{Ad} T_j (T_i)} T_i^{-1} = T_j$$

$$\Longleftarrow \mathrm{Ad}\, T_i\, \mathrm{Ad}\, T_j (T_i) = T_j$$

Th 1 $\Rightarrow \mathrm{Ad}\, T_i\, \mathrm{Ad}\, T_j (f_i) = f_j$

$\mathrm{Ad}\, T_i\, \mathrm{Ad}\, T_j (e_i) = e_j$　similarly

$\mathrm{Ad}'T_i\,\mathrm{Ad}'T_j(t_i) = t_j$ easy to check

Recall $'T_i = \exp_{\xi_i^{-1}}(\xi_i^{-1} e_i t_i^{-1}) \cdots\cdots \xi_i^{\varepsilon_i(l_i+1)/2}$

↑

Apply $\mathrm{Ad}'T_i\,\mathrm{Ad}'T_j$ to

$$\overset{\xi_i=\xi_j}{\Longrightarrow} \mathrm{Ad}'T_i\,\mathrm{Ad}'T_j('T_i) = 'T_j \quad \text{OK}$$

Other cases similar

(↖ do not need $\xi_i = \xi_j$)

↑

$\mathrm{Ad}'T_i \cdots (f_i) = f_i$

//

$$\begin{tikzcd} & B_W \arrow[r] & \operatorname{End} V \text{ or } \operatorname{Aut}(U_\xi^-) \\ & \sigma_i \arrow[r, mapsto] & 'T_i \qquad \mathrm{Ad}'T_i \\ & \tilde{w} \arrow[r, mapsto, "\text{group hom.}"'] & 'T_w \\ W \arrow[uuur, "\text{section by Prop 2}"] & & \\ w \arrow[uur, mapsto] & & \end{tikzcd}$$

∴

↖ well-defined

$$'T_w\,'T_{w'} = 'T_{ww'}$$

if $l(ww') = l(w) + l(w')$

§ PBW bases

Prop (1) $w \in W$ $\quad w(\alpha_i) \in \Delta^+ \Longrightarrow \underset{\nearrow}{T_w(f_i)} \in U_q^-$

analog of a root vector for $w(\alpha_i)$

(2) $w(\alpha_i) = \alpha_k \quad \Longrightarrow \quad T_w(f_i) = f_k$

(proof) Case 1°. rk 2 case $\quad \mathfrak{g} = \langle i, j \rangle$

1,1° $a_{ij} = -1 = a_{ji}$

- $s_i(\alpha_j) = \alpha_i + \alpha_j$
- $s_i s_j(\alpha_i) = \alpha_j$

Need to consider

$T_i(f_j) = \mathrm{ad}^*(f_i)(f_j) \in U_q^-$

explicit expr.

$T_i T_j(f_i) = f_j$ by Th 1

1,2° other cases $\longrightarrow$ Exercise

Case 2° general case

reduction to case 1°